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Thermodynamics of specific modified gravity with generalized interaction term

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We investigate the generalized second law of thermodynamics by assuming the interaction of dark energy and dark matter in Chern–Simons modified gravity. We consider a family of holographic dark energy models by assuming its various cutoffs such as Hubble horizon, event horizon, their combination, Ricci scalar and its generalized form. The general criteria of generalized second law of thermodynamics in terms of coincidence parameter is being developed. This criterion is being applied for the above-mentioned holographic dark energy models to check the validity of the generalized second law of thermodynamics and the constraints where it is respected are referred.

Keywords: Generalized second law of thermodynamics; holographic dark energy models; generalized interaction term; dynamical Chern–Simons modified gravity.

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1. Introduction

The phase transition of the universe has been suggested by many observations like Ia Supernova [1, 2], cosmic microwave background [3] and other recent data [4]. The phenomenon responsible for this phase transition is an unknown component called dark energy (DE) (possesses repulsive force) [5] which comprises about 73% of the total universe. Various DE models have been presented for explaining the DE phenomenon such as family of Chaplygin gas [6], holographic [7, 8], new agegraphic [9], pilgrim [10] DE models, etc. Holographic DE (HDE) model is one of them which has been developed on the basis of holographic principle [11] while HDE density has been generated on the basis of Cohen *et al.* [11] relation. It is defined as the vacuum

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energy of a system with specific size whose maximum amount should not exceed the black hole mass with the same size which can be elaborated in mathematical form as $L^3\rho_\Lambda \leq LM_p^2$, where $M_p^2 = (8\pi G)^{-1}$ is the reduced Planck mass and L represents the infra-red (IR) cutoff or length scale parameter. Through this bound, Li [8] proposed the constraint on the DE density

$$\rho_\Lambda = 3\zeta^2 M_p^2 L^{-2}, \tag{1}$$

where ζ is the dimensionless HDE constant parameter.

The interesting feature of HDE density is that it provides a relation between ultraviolet (bound of vacuum energy density) and IR (size of the universe) cutoffs. However, a controversy about the selection of IR cutoff of HDE has been raised since its birth. As a result, different people have suggested different expressions. For example, $L = H^{-1}$ (where H is the Hubble parameter) [8], $L = R^{-1/2} = (2H^2 + \dot{H} + \frac{k}{a^2})^{-1/2}$ (Ricci scalar) (the corresponding energy density known as Ricci dark energy (RDE)) [12], $L = (\alpha H^2 + \beta\dot{H})^{-1/2}$ (the corresponding energy density known as new HDE (NHDE)), where α and β are constants [13]. A linear combination of apparent and event horizon has been proposed [14].

Moreover, the generalized second law of thermodynamics (GSLT) has been discussed extensively in the scenario of accelerated expansion of the universe. Many people have explored the validity of GSLT of different systems including interacting of two components of fluid like DE and dark matter (DM) [15] and interacting three components of fluid [16] in the FRW universe by using simple horizon entropy of the universe. Sharif and his collaborators [17] have discussed GSLT in the interacting scenario of modified holographic DE in flat and nonflat Kaluza–Klein universe with the help of simple horizon entropy of the universe. The logarithmic and power law corrections to the usual entropy-area relation of the expanding universe may be useful in exploring the GSLT.

Moreover, several modified theories of gravity are $f(R)$, $f(T)$ [18], $f(R,\mathcal{T})$ [19], $f(G)$ [20], $f(T,T_G)$ [21], $f(T,\mathcal{T})$ [22, 23] (where R is the curvature scalar, T denotes the torsion scalar, $\mathcal{T}$ is the trace of the energy–momentum tensor and G is the invariant of Gauss–Bonnet defined as $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$). For clear review of DE models and modified theories of gravity, see [24]. However, Chern–Simons modified gravity has been developed recently [25] and was not a random extension. This gravity is motivated from string theory (as a necessary anomaly-canceling term to conserve unitarity [26] as well as from loop quantum gravity [27]. This modified gravity exhibits the violation of parity symmetry in Einstein–Hilbert action because of the inclusion of the Pontryagin density (a topological term in four dimensions, unless the coupling constant is not constant or promoted to a scalar field). The detail of this modified gravity has been given in [28]. Many authors have investigated cosmic acceleration in the dynamical Chern–Simons modified gravity by taking various HDE models [29–33].

We have also investigated the GSLT by assuming interacting scalar field dark energy model (possessing a noncanonical kinetic term) with DM at the apparent horizon in the framework of modified Chern–Simons gravity [34]. We also considered the usual entropy and its corrected forms (logarithmic and power law) in this work. Recently, Arevalo *et al.* [35] have investigated the GSLT in detail for interacting scenario of DE (HDE and RDE models) and DM in flat FRW universe. In the present work, we extend this work in Chern–Simons modified gravity and check the validity of GSLT in various scenarios of HDE and RDE models. The paper is organized as follows: Next section contains the basic equations of modified gravity and interacting DE and DM. Sections 3 and 4 contain the discussion of general framework of GSLT and the presence of HDE models, respectively. Sections 5 and 6 are devoted to the study of GSLT by assuming RDE and NHDE, respectively. We summarized our results in the last section.

2. Basic Equations

The modified Chern–Simons gravity is motivated from string theory of low energy limit which comprises a necessary anomaly-canceling for correction of Einstein–Hilbert action and for describing modified Chern–Simons gravity as an effective theory. The action which describes the Chern–Simons theory is defined as

$$S = \frac{1}{16\pi G}\int d^4x\left[\sqrt{-g}R + \frac{\ell}{4}\theta^{\star}R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\theta\nabla_\nu\theta + V(\theta)\right] + S_{\mathrm{mat}}. \tag{2}$$

Here, $R, {}^{\star}R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu}, \ell, \theta, S_{\mathrm{mat}}$ and $V(\theta)$ appear as Ricci scalar, a topological invariant called the Pontryagin term, coupling constant, dynamical variable, action of matter and the potential, respectively. We set $V(\theta) = 0$ for simplicity. By varying the above action according to the metric $g_{\mu\nu}$ as well as the scalar field θ, we obtain the following field equations

$$G_{\mu\nu} + \ell C_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad g^{\mu\nu}\nabla_\mu\nabla_\nu\theta = -\frac{\ell}{64\pi}{}^{\star}R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu}, \tag{3}$$

respectively. In these equations, $G_{\mu\nu}$ and $C_{\mu\nu}$ are the Einstein and Cotton tensors, respectively. The Cotton tensor is defined as follows:

$$C^{\mu\nu} = -\frac{1}{2\sqrt{-g}}((\nabla_\rho\theta)\varepsilon^{\rho\beta\tau(\mu}\nabla_\tau R^{\nu)}_{\beta}) + (\nabla_\sigma\nabla_\rho\theta)^{\star}R^{\rho(\mu\nu)\sigma}. \tag{4}$$

In this framework, the energy–momentum tensors have the following forms:

$$\hat{T}^{\theta}_{\mu\nu} = \nabla_\mu\theta\nabla_\nu\theta - \frac{1}{2}g_{\mu\nu}\nabla^\rho\theta\nabla_\rho\theta, \quad T_{\mu\nu} = (\rho\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{5}$$

where $T^{\theta}_{\mu\nu}$ corresponds to scalar field contribution while $\hat{T}_{\mu\nu}$ represents the DE and CDM contributions. Also, ρ represents the energy density due to DE and CDM, while p represents the pressure due to only the DE component. Moreover, $U_\mu = (1, 0, 0, 0)$ is the four velocity. Using Eqs. (3) and (5), we get the following Friedmann equation for flat universe

$$H^2 = \frac{1}{3}(\rho_{\mathrm{dm}} + \rho_{\mathrm{de}}) + \frac{1}{6}\dot{\theta}^2, \tag{6}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, dot denotes the derivative of scale factor a with respect to cosmic time and $m_{pl}^{-1} = 8\pi G = 1$.

The field equation (3) is associated with the scalar field and ${}^{\star}R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu} = 0$ for FRW metric. In this scenario, Eq. (3) takes the form

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\theta = g^{\mu\nu}[\partial_{\mu}\partial_{\nu}\theta - \Gamma^{\rho}_{\mu\nu}\partial_{\rho}\theta] = 0. \tag{7}$$

By assuming $\theta = \theta(t)$, we can obtain the following equation:

$$\ddot{\theta} + 3H\dot{\theta} = 0 \Rightarrow \dot{\theta} = ba^{-3}, \tag{8}$$

where b is an integration constant. In this way, Eq. (6) takes the form

$$3H^2 = \rho_{\rm de} + \rho_{\rm dm} + \frac{b^2}{2}a^{-6}, \tag{9}$$

$$2\dot{H} + 3H^2 = -(p_{\rm de} + p_{\rm dm}), \tag{10}$$

where $(\rho_{\rm dm}, p_{\rm dm})$ and $(\rho_{\rm de}, \rho_{\rm de})$ are energy densities and pressures due to DM and DE. By rearranging Eq. (10), we get

$$\dot{H} = -\frac{1}{2}\left(\rho_{\rm de} + \rho_{\rm dm} + p_{\rm de} + p_{\rm dm} + \frac{b^2}{2}a^{-6}\right). \tag{11}$$

The total energy density is $\rho_{\rm tot} = \rho_{\rm dm} + \rho_{\rm de}$. In our model, assuming that DE component interacts with DM component described by phenomenological coupling function (D), the energy conservation equation is separated as

$$\dot{\rho}_{\rm de} + 3H(\rho_{\rm de} + p_{\rm de}) = -D, \tag{12}$$

$$\dot{\rho}_{\rm dm} + 3H(\rho_{\rm dm} + p_{\rm dm}) = D, \tag{13}$$

where D is the interaction term. Several interacting models have been considered with interacting term as the functions of energy densities and the Hubble parameter [36]. In the present scenario, we will utilize the interaction term (D) in general [37]. We assume auxiliary variable as $r(a)$ (coincidence parameter) which is defined as

$$r = \frac{\rho_{\rm dm}}{\rho_{\rm de}}, \tag{14}$$

where the coincidence parameter is used to evaluate the cosmic coincidence problem. The energy density $\rho_{\rm de}$ can be written as $\rho_{\rm de} = \frac{3H^2 - 1/2b^2a^{-6}}{1+r}$. Taking the derivative of $\rho_{\rm de}$, we obtain

$$\dot{\rho}_{\rm de} = \frac{1}{(1+r)^2}\bigg(-3H(1+r)(\rho_{\rm de} + \rho_{\rm dm} + p_{\rm de} + p_{\rm dm}) - 3H^3r' + \frac{3H}{2}(1+r) \times b^2a^{-6} + \frac{b^2}{2}a^{-6}Hr'\bigg),$$

where $\dot{r} = Hr'$. By inserting the value of $\dot{H}$, we get

$$\dot{\rho}_{\rm de} = \frac{-3(1+r)H\left[\rho_{\rm de} + \rho_{\rm dm} + p_{\rm de} + p_{\rm dm} + \frac{b^2}{2}a^{-6}\right] - 3\dot{r}H^2}{(1+r)^2}, \tag{15}$$

After replacing the $\dot{\rho}_{\rm de}$ in Eq. (13), we can obtain the interaction parameter as follows [38]:

$$\frac{D}{3H^3} = \frac{1}{(1+r)^2}\left[r' - (rp_{\rm de} - p_{\rm dm})\frac{(r+1)}{H^2} - \frac{b^2a^{-6}}{2H^2} - \frac{b^2a^{-6}r'}{6H^2}\right], \tag{16}$$

where D depends upon $(H, r, \rho_{\rm de}, p_{\rm dm}, r', a, b)$, while prime denotes the derivative with respect to $x = \ln a$. For positivity of the above function, we require

$$r' \geq \frac{(rp_{\rm de} - p_{\rm dm})(r+1)}{H^2} + \frac{b^2a^{-6}}{2H^2}\left(1 + \frac{r'}{3}\right). \tag{17}$$

3. Thermodynamics

Here, we investigate the validity of GSLT for interacting DE and DM cosmological scenarios. According to this law, the sum of entropy of matter inside and at the horizon remains always positive with the passage of time [39]. Thermodynamics of black hole plays a crucial rule for thermodynamical interpretation of the universe. In view of the proportionality relation between entropy of black hole horizon and horizon area, Bekenstein [40] suggested that the sum of black hole entropy and the background entropy must be an increasing quantity with time. The first law of thermodynamics gives

$$TdS = pdV + dE, \tag{18}$$

where T, S, E and p represent temperature, entropy, internal energy and pressure of the system, respectively. Splitting this law for DE, DM and differentiating with respect to time, we obtain [40]

$$T_{\rm de}dS_{\rm de} = p_{\rm de}dV + dE_{\rm de}, \tag{19}$$

$$T_{\rm dm}dS_{\rm dm} = p_{\rm dm}dV + dE_{\rm dm}, \tag{20}$$

where ($S_{\rm de}$, $S_{\rm dm}$) denote the entropies of DE and DM components and ($E_{\rm de}$, $E_{\rm dm}$) are the corresponding energies, respectively. However, the volume of the system is $V = \frac{4\pi L^3}{3}$ and the energy density with the scale length (L) can be defined as

$$E_{\rm de} \equiv \frac{4\pi}{3}L^3\rho_{\rm de}, \quad E_{\rm dm} \equiv \frac{4\pi}{3}L^3\rho_{\rm dm}, \tag{21}$$

which leads to

$$\dot{S}_{\rm de} = \frac{4\pi L^2\dot{L}}{T_{\rm de}}(\rho_{\rm de} + p_{\rm de}) + \frac{4\pi}{3T_{\rm de}}L^3\dot{\rho}_{\rm de}, \tag{22}$$

$$\dot{S}_{\rm dm} = \frac{4\pi L^2\dot{L}}{T_{\rm dm}}(\rho_{\rm dm} + p_{\rm dm}) + \frac{4\pi}{3T_{\rm dm}}L^3\dot{\rho}_{\rm dm}. \tag{23}$$

These equations yield

$$\dot{S}_{\rm de} + \dot{S}_{\rm dm} = \frac{4\pi L^2}{T_{de}}\left(\dot{L}(\rho_{\rm de} + p_{\rm de}) + \frac{L}{3}\dot{\rho}_{\rm de}\right) + \frac{4\pi L^2}{T_{\rm dm}}\left(\dot{L}(\rho_{\rm dm} + p_{\rm dm}) + \frac{L}{3}\dot{\rho}_{\rm dm}\right). \tag{24}$$

In the present case, we consider the thermal equilibrium condition, i.e. $T_{\rm de} \approx T_{\rm dm} \approx T$. Hence, the above equation leads to

$$\dot{S}_{\rm de} + \dot{S}_{\rm dm} = \frac{4\pi L^2}{T}(\rho_{\rm de} + \rho_{\rm dm} + p_{\rm de} + p_{\rm dm})\dot{L} + \frac{4\pi}{3T}(\dot{\rho}_{\rm de} + \dot{\rho}_{\rm dm})L^3.$$

Inserting the value of $\dot{H}$ in above equation, we will get

$$\dot{S}_{\rm de} + \dot{S}_{\rm dm} = \frac{4\pi}{T}L^2(\rho_{\rm de} + \rho_{\rm dm} + p_{\rm de} + p_{\rm dm})(\dot{L} - HL). \tag{25}$$

After some calculations, the above equation reduces to

$$\dot{S}_{\rm de} + \dot{S}_{\rm dm} = -\frac{8\pi}{T}L^2\dot{H}(\dot{L} - HL) - \frac{2\pi}{T}L^2b^2a^{-6}(\dot{L} - HL). \tag{26}$$

Here, we assume the temperature of the fluid T is the same as the horizon temperature, i.e. $T = T_H$. The horizon temperature can be defined as $T_H = \frac{1}{2\pi L}$ [41]. The entropy associated with this horizon is $S_H = 8\pi^2L^2$. Hence, the rate of change of total entropy turns out to be

$$\begin{aligned}\dot{S}_{\rm tot} &= \dot{S}_{\rm de} + \dot{S}_{\rm dm} + \dot{S}_H,\\ &= 16\pi^2L\dot{L} - 16\pi^2L^3(\dot{L} - HL)\dot{H} - 4\pi^2L^3(\dot{L} - HL)b^2a^{-6}.\end{aligned} \tag{27}$$

By simplifying the above equation, we obtain the entropy change of interaction system as

$$\Rightarrow \frac{\dot{S}_{\rm tot}}{16\pi^2L} = (1 - \dot{H}L^2)(\dot{L} - HL) + HL - \frac{1}{4}L^2b^2(\dot{H} - LH)a^{-6}. \tag{28}$$

Thus, total entropy in Eq. (28) depends on the parameters such as (H, L, $\dot{H}$, $\dot{L}$, a, b). Using Eq. (11) for $\dot{H}$ into Eq. (28), we get

$$\frac{\dot{S}_{\rm tot}}{8\pi^2H^2L^3} = \left(3 + \frac{p}{H^2}\right)(\dot{L} - HL) + \frac{2\dot{L}}{H^2L^2} - \frac{b^2a^{-6}}{H^2}(\dot{L} - HL). \tag{29}$$

This is the general expression of GSLT which depends on $(L,\ H,\ p,\ \dot{L})$, where the scale length $\dot{L}$ and the pressure (p) could determine the sign change of entropy. Next, we will consider some examples of HDE models for evaluating the validity of GSLT.

4. Thermodynamics of HDE Models

Nojiri and Odintsov [42] have evaluated the early-time and late-time universe by using unification approach based on phantom cosmology. They considered the gravity-scalar system which contains usual potential and scalar coupling function in front of kinetic term and investigated the possibility of phantom–non-phantom transition appearing in such a way that the universe could have effective phantom equation of state at early time as well as at late time. Also, they suggested generalized holographic dark energy and suggested various possibilities in order to solve the coincidence problem, crossing of phantom divide and unification of early-time inflationary and late-time accelerating phantom universe. The bound for holographic entropy which decreases in phantom era is also discussed.

Recently, they [43] also considered the generalized holographic dark energy model where the infrared cutoff is identified with the combination of the FRW universe parameters: the Hubble rate, particle and future horizons, cosmological constant, the universe lifetime (if finite) and their derivatives. They demonstrated that with the corresponding choice of the cutoff one can map such holographic dark energy to modified gravity or gravity with a general fluid. Explicitly, $F(R)$ gravity and the general perfect fluid are worked out in detail and the corresponding infrared cutoff is found. Using this correspondence, we get realistic inflation or viable dark energy or a unified inflationary-dark energy universe in terms of covariant holographic dark energy.

In above works, the authors have made versatile study of cosmic acceleration in the presence of generalized HDE and specific modified gravity and pointed out various possibilities of solving longstanding problem like coincidence and other cosmic aspects. However, we have chosen specific forms of HDE and discuss the GSLT in view of coincidence parameter r in dynamical Chern–Simons modified gravity.

Firstly, we consider the HDE for evaluating the GSLT and the energy density is given by [44]

$$\rho_{\rm de} = \frac{3c^2}{L^2}, \tag{30}$$

where c is the positive constant [45, 46]. In this scenario, the coincidence parameter turns out to be

$$r = \frac{H^2L^2}{c^2}\left(1 - \frac{b^2a^{-6}}{6H^2}\right) - 1. \tag{31}$$

One can obtain the derivative of length-scale parameter in terms of cosmic coincidence parameter as follows:

$$\dot{L} = \frac{c^2r'H}{2L\left(H^2 - \frac{b^2a^{-6}}{6}\right)} - \frac{c^2(1+r)H\dot{H}}{LH\left(1 - \frac{b^2a^{-6}}{6H^2}\right)^2} - \frac{c^2(1+r)b^2a^{-6}H}{2L\left(H^2 - \frac{b^2a^{-6}}{6}\right)^2}, \tag{32}$$

which leads to

$$\dot{L} - HL = \frac{1}{LH}\left(\frac{c^2r'}{2\left(1 - \frac{b^2a^{-6}}{6H^2}\right)} - \frac{c^2(1+r)\dot{H}}{H^2\left(1 - \frac{b^2a^{-6}}{6H^2}\right)^2} - \frac{c^2(1+r)b^2a^{-6}}{2H^2\left(1 - \frac{b^2a^{-6}}{6H^2}\right)^2} - \frac{c^2(1+r)}{1 - \frac{b^2a^{-6}}{6H^2}}\right). \tag{33}$$

Hence, the expression of GSLT turns out to be

$$\begin{aligned}\frac{\dot{S}_{\rm tot}}{16\pi^2 L} &= (1+H^2L^2(1+q))\frac{c^2}{LH}\\&\quad\times\left[\frac{r'}{2\left(1-\frac{b^2a^{-6}}{6H^2}\right)}+\frac{(1+r)(1+q)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{(1+r)b^2a^{-6}}{2H^2\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{1+r}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}\right]\\&\quad+HL-\frac{b^2L^2}{4}\frac{c^2}{LH}\\&\quad\times\left[\frac{r'}{2\left(1-\frac{b^2a^{-6}}{6H^2}\right)}+\frac{(1+r)(1+q)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{(1+r)b^2a^{-6}}{2H^2\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{1+r}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}\right].\end{aligned}$$

Here, we have inserted the values of $\dot{H}$ in terms of deceleration parameter, i.e. $q=-\frac{\dot{H}}{H^2}-1$, where sign is considered to be negative for explaining the accelerated expansion of the universe. By rearranging the above equation, we get

$$\begin{aligned}\frac{H\dot{S}_{\rm tot}}{16\pi^2 c^2} &= \left(1+\frac{c^2(1+r)(1+q)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}\right)\\&\quad\times\left[\frac{r'}{2\left(1-\frac{b^2a^{-6}}{6H^2}\right)}+\frac{(1+r)(1+q)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{(1+r)b^2a^{-6}}{2H^2\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{1+r}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}\right]\\&\quad+\frac{(1+r)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}-\frac{b^2L^2}{4}\left[\frac{r'}{2\left(1-\frac{b^2a^{-6}}{6H^2}\right)}+\frac{(1+r)(1+q)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}\right.\\&\quad\left.-\frac{(1+r)b^2a^{-6}}{2H^2\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{1+r}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}\right].\end{aligned}\tag{34}$$

Focusing on the relation between change in entropy and interaction, we take the derivative of Eq. (30) and using it in Eq. (12), we get

$$\dot{L}=\frac{HL}{2}\left[3+\frac{L^2}{c^2}\left(p_{\rm de}+\frac{D}{3H}\right)\right].\tag{35}$$

Replace the value of $\dot{L}$ in Eq. (29) as

$$\begin{aligned}\frac{\dot{S}_{\rm tot}}{8\pi^2L^3H^2} &= \left(3+\frac{p_{\rm de}+p_{\rm dm}}{H^2}\right)\left[\frac{HL}{2}\left(3+\frac{L^2}{c^2}\left(p_{\rm de}+\frac{D}{3H}\right)-HL\right)\right]\\&\quad+\frac{2}{H^2L^2}\left[\frac{HL}{2}\left(3+\frac{L^2}{c^2}\left(p_{\rm de}+\frac{D}{3H}\right)\right)\right]\\&\quad-\frac{b^2a^{-6}}{H^2}\left(\frac{HL}{2}\left(3+\frac{L^2}{c^2}\left(p_{\rm de}+\frac{D}{3H}\right)\right)-HL\right).\end{aligned}$$

For the pressureless DM component ($p_{\rm dm}=0$), we get

$$\frac{S'_{\rm tot}}{4\pi^2L^2}=4+\left(2+3H^2L^2+p_{\rm de}L^2\right)\left[1+\frac{L^2}{3c^2}\left(3p_{\rm de}+\frac{D}{H}\right)\right]$$
$$-b^2a^{-6}L^2\left(1+\frac{L^2}{c^2}\left(p_{\rm de}+\frac{D}{3H}\right)\right).$$

By rearranging the above equation, we get

$$\frac{S'_{\rm tot}}{4\pi^2L^2}=4+\left[2+H^2L^2\left(3+\frac{p_{\rm de}}{H^2}\right)\right]\left[1+\frac{L^2}{c^2}\left(p_{\rm de}+\frac{D}{3H}\right)\right]$$
$$-b^2a^{-6}\left(1+\frac{L^2}{c^2}\left(p_{\rm de}+\frac{D}{3H}\right)\right).$$

The total entropy change in terms of cosmic coincidence r can be written as

$$\frac{S'_{\rm tot}}{4\pi^2L^2}=4+\left(2+\frac{c^2(1+r)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}(3+P_{\rm de})\right)\left[1+\frac{(1+r)}{1-\frac{b^2a^{-6}}{6H^2}}(P_{\rm de}+\kappa)\right]$$
$$-b^2a^{-6}L^2\left[1+\frac{(1+r)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}(P_{\rm de}+\kappa)\right].$$

Thus, the total entropy function is a function of $(r,\ P_{\rm de},\ \kappa)$, where $P_{\rm de}=\frac{p_{\rm de}}{H^2}$, and $\kappa=\frac{D}{3H}$. Hence, the GSLT can be satisfied if

$$r\leq-1-\frac{1-\frac{b^2a^{-6}}{6H^2}}{P_{\rm de}+\kappa}.$$

Next, we utilize various forms of length scale parameter for analyzing the validity of GSLT.

4.1. *GSLT of HDE with apparent horizon*

Here, we will discuss the GSLT by assuming different versions of horizons of accelerating universe such as apparent horizon $L_{\rm app}$, event horizon $L_{\rm event}$ and the linear combination of both. The apparent horizon $L_{\rm app}$ for flat FRW universe can be defined as

$$L\equiv L_{\rm app}\equiv H^{-1}. \tag{36}$$

For this scale length, DE density turns out to be

$$\rho_{\rm de}=3c^2H^2.$$

Taking the derivative of Eq. (36), we obtain

$$\dot{L}_{\rm app}=-\frac{\dot{H}}{H^2}. \tag{37}$$

Utilizing the derivative of $L_{\rm app}$ in Eq. (28), we obtain

$$H\frac{\dot{S}_{\rm app}}{16\pi^2}=\left(\frac{\dot{H}}{H^2}\right)^2+\frac{b^2a^{-6}}{4H^2}\left(\frac{\dot{H}}{H^2}+1\right). \tag{38}$$

This equation shows that GSLT remains valid throughout the region. For apparent horizon, the cosmic coincidence parameter turns out to be

$$r_{\rm H}=\frac{1}{c^2}\left(1-\frac{1}{6H^2}b^2a^{-6}\right)-1,$$

which remains positive for $c^2<1-\frac{b^2a^{-6}}{6H^2}$.

4.2. *GSLT on HDE with event horizon*

The event horizon, $L_{\rm event}$, can be defined as

$$L_{\rm event}=a\int_t^\infty\frac{dt}{a}=a\int_t^\infty\frac{da}{Ha^2}\Rightarrow\dot{L}_{\rm event}=HL_{\rm event}-1. \tag{39}$$

By inserting the event horizon and its derivative in Eq. (28), we get the expression which depends on $(H,\ L_{\rm event},\dot{H})$ as follows:

$$\frac{\dot{S}_{\rm event}}{16\pi^2L}=-1+\dot{H}L_{\rm event}^2+HL_{\rm event}+\frac{1}{4}L_{\rm event}^2b^2a^{-6}. \tag{40}$$

By using Eqs. (30) and (40), we obtain the GSLT expression in terms of coincidence parameter

$$\frac{\dot{S}_{\rm event}}{16\pi^2L_{\rm event}}=\left(-1+\frac{c^2}{2}r'-c^2(1+r)\right)+c\sqrt{(1+r)\left(1-\frac{b^2a^{-6}}{H^2}\right)}$$
$$+c\sqrt{\frac{(1+r)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}}-\frac{1}{4}L_{\rm event}^2b^2a^{-6}.$$

From the above expression, it can be noted that GSLT depends upon the cosmic coincidence parameter and Chern–Simons modified gravity parameter (b^2). However, if r behaves like increasing function over late times, then r' is a positive function. For this scenario, GSLT is respected along the conditions $0<b^2\ll1$ and $r<-1-1/c^2$.

4.3. *GSLT on HDE with linear combination of apparent and event horizons*

Now, we consider a linear combination of the apparent horizon $L_{\rm app}$ and event horizon $L_{\rm event}$ as considered in [14] while investigating thermodynamics and phantom barrier. This is given as

$$L_{\rm ae}=\lambda_{\rm app}L_{\rm app}+\lambda_{\rm event}L_{\rm event}, \tag{41}$$

where $L_{\rm app}$ and $L_{\rm event}$ are defined in Eqs. (36) and (39), respectively, and both $\lambda_{\rm app}$ and $\lambda_{\rm event}$ are positive constants. The derivative of $L_{\rm ae}$ with respect to time becomes $\dot{L}_{\rm ae} = HL_{ae} + \lambda_{\rm app} - \lambda_{\rm event}$. This expression is not always positive, as can be retrieved from literature comparing it with Eq. (33), where we conclude that $\lambda_{\rm app} < \lambda_{\rm event}$, thus at some points the term $\dot{L}_{ae}$ may be negative behavior. Hence, for this horizon, GSLT becomes

$$\frac{\dot{S}_{\rm ae}}{16\pi^2 L_{\rm ae}} = (1 - \dot{H}L_{\rm ae}^2)(\lambda_{\rm app} - \lambda_{\rm event}) + HL_{\rm ae} - \frac{1}{4}b^2(\lambda_{\rm app} - \lambda_{\rm event})L_{\rm ae}^2 a^{-6}. \tag{42}$$

For this scale length, the total entropy change in terms of cosmic coincidence turns out to be

$$\frac{\dot{S}_{\rm ae}}{16\pi^2 L_{\rm ae}} = \left(1 - \frac{c^2 r'}{2} - c^2(1+r) - b^2 a^{-6} H - \frac{b^2}{4}L_{\rm ae}^2 a^{-6}\right)(\lambda_{\rm app} - \lambda_{\rm event}) - \left[\sqrt{1 - \frac{b^2 a^{-6}}{6H^2}}(\lambda_{\rm app} - \lambda_{\rm event})^2 + \frac{1}{\sqrt{1 - \frac{b^2 a^{-6}}{6H^2}}}\right] c\sqrt{1+r}.$$

The GSLT can be respected by the evolution of the parameters. If GSLT is respected, then the right side of the above equation must be positive, which leads to inequalities in terms of the interaction according to the sign of each individual term. Thus $\dot{S}_{\rm ae} \geq 0$ if $\lambda_{\rm app} \geq (\frac{c^2 r'}{2} + c^2(1+r) + b^2 a^{-6} H + \frac{b^2}{4} L^2 a^{-6})(\lambda_{\rm app} - \lambda_{\rm event}) + [\sqrt{1 - \frac{b^2 a^{-6}}{6H^2}}(\lambda_{\rm app} - \lambda_{\rm event})^2 + \frac{1}{\sqrt{1-\frac{b^2 a^{-6}}{6H^2}}}]c\sqrt{1+r} + \lambda_{\rm event}$ for $6H^2 \geq b^2 a^{-6}$.

5. GSLT on RDE and NHDE

Here, we assume the Ricci DE model and explore the validity of GSLT. The Ricci dark energy density can be defined as

$$\rho_{\rm de} = 3c^2(2H^2 + \dot{H}), \tag{43}$$

where c^2 is a positive constant. For this DE density, the dark energy pressure leads to

$$p_{\rm de} = H^2 - \frac{2}{3c^2}\rho_{\rm de}. \tag{44}$$

By using Eqs. (30) and (44), we obtain

$$\dot{L} = \frac{HL}{2c^2}(3c^2 - 2 + H^2 L^2(1+\kappa)). \tag{45}$$

By utilizing the expressions (11), (28), (29), (30) and (45), we can obtain the expression of GSLT for Ricci-like DE in terms of cosmic coincidence parameter as

$$\frac{\dot{S}_{\rm R}}{16\pi^2 L} = \frac{HL}{2}\left[\frac{2c^2(1+r)}{\left(1 - \frac{b^2 a^{-6}}{6H^2}\right)}\left(1 - \frac{2}{c^2} + \frac{c(1+r)(1+\kappa)}{\left(1 - \frac{b^2 a^{-6}}{6H^2}\right)}\right)\right] + 2 - \frac{1}{4}L^2 b^2\left[1 - \frac{2}{c^2} + \frac{c(1+r)(1+\kappa)}{1 - \frac{b^2 a^{-6}}{6H^2}}\right]a^{-6}.$$

By inserting the value of L, we obtain

$$\frac{\dot{H}\dot{S}_{\rm R}}{8\pi^2 c^2\left(1-\frac{b^2a^{-6}}{6H^2}\right)} = \frac{2c^2(1+r)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)}\left(1+\frac{c(1+r)(1+\kappa)}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)} - \frac{4}{1-\frac{b^2a^{-6}}{6H^2}}\right) + 2 - \frac{1}{4}\left(1-\frac{2}{c^2}+\frac{c(1+r)(1+\kappa)}{1-\frac{b^2a^{-6}}{6H^2}}\right).$$

The above equation depends on (r,κ,b,a). It is noted that there are some possibilities for which $\dot{S}_{\rm R}$ is respected, i.e. $(\frac{HL(1+r)c^2}{1-\frac{b^2a^{-6}}{H^2}})N + 2 \geq \frac{1}{4}L^2b^2a^{-6}N$ with $6H^2 \geq b^2a^{-6}$, where $N = 1-\frac{2}{c^2}+\frac{c(1+r)(1+\kappa)}{1-\frac{b^2a^{-6}}{6H^2}}$.

Now, explicitly, we assume the entropy change without the interaction. Furthermore from Eqs. (30), (44) and (35), we obtain $L^2 = \frac{c^2(1+r)}{1-\frac{b^2a^{-6}}{6H^2}}$. Taking the derivative of L, we get

$$\dot{L} = \frac{c^2 r'}{2L\left(H^2-\frac{b^2a^{-6}}{6}\right)} - \frac{c^2(1+r)H\dot{H}}{L\left(H^2-\frac{b^2a^{-6}}{6}\right)^2} - \frac{c^2(1+r)b^2a^{-6}H}{2L\left(H^2-\frac{b^2a^{-6}}{6}\right)^2}. \tag{46}$$

From Eq. (11), $\dot{H}$ can be obtained as follows:

$$\dot{H} = -\frac{1}{2}\left((r+1)\frac{3c^2}{L^2}+H^2-\frac{2}{3}\rho_{\rm de}+\frac{1}{2}b^2a^{-6}\right). \tag{47}$$

Substituting the value of $\dot{H}$ and $\dot{L}$ in (28), we get

$$\begin{aligned}\frac{H\dot{S}_{\rm R}}{16\pi^2} &= \left(1+\frac{3c^2(1+r)}{2}+\frac{H^2}{L^2}-2+\frac{1}{2}L^2b^2a^{-6}\right)\\ &\quad\times\left(\frac{c^2r'}{1-\frac{b^2a^{-6}}{6H^2}}+\frac{c^2(1+r)\left(\frac{3c^2(1+r)}{2L^2}+\frac{H^2}{2}-\frac{1}{L^2}+\frac{1}{4}b^2a^{-6}\right)}{H^2\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}\right.\\ &\quad\left.-\frac{c^2(1+r)b^2a^{-6}}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{c^2(1+r)}{1-\frac{b^2a^{-6}}{6H^2}}\right)+\frac{c^2(1+r)}{1-\frac{b^2a^{-6}}{6H^2}}\\ &\quad-\frac{1}{4}L^2\left(\frac{c^2r'}{1-\frac{b^2a^{-6}}{6H^2}}-\frac{c^2(1+r)b^2a^{-6}}{\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}\right.\\ &\quad\left.+\frac{c^2(1+r)\left(\frac{3c^2(1+r)}{2L^2}+\frac{H^2}{2}-\frac{1}{L^2}+\frac{1}{4}b^2a^{-6}\right)}{H^2\left(1-\frac{b^2a^{-6}}{6H^2}\right)^2}-\frac{c^2(1+r)}{1-\frac{b^2a^{-6}}{6H^2}}\right).\end{aligned}$$

This relation depends upon $(r,\ r')$. We introduce two auxiliary parameters such as $G = 1 + \frac{3c^2(1+r)}{2} + \frac{H^2}{L^2} - 2 + \frac{1}{2}L^2b^2a^{-6}$, $F = (\frac{c^2r'}{1-\frac{b^2a^{-6}}{6H^2}} - \frac{c^2(1+r)b^2a^{-6}}{(1-\frac{b^2a^{-6}}{6H^2})^2} + \frac{c^2(1+r)(\frac{3c^2(1+r)}{2L^2}+\frac{H^2}{2}-\frac{1}{L^2}+\frac{1}{4}b^2a^{-6})}{H^2(1-\frac{b^2a^{-6}}{6H^2})^2} - \frac{c^2(1+r)}{1-\frac{b^2a^{-6}}{6H^2}})$. It is observed that GSLT is satisfied when both parameters must be positive with condition $G + \frac{c^2(1+r)}{H(1-\frac{b^2a^{-6}}{6H^2})} \geq \frac{1}{4}L^2$ for $6H^2 \geq b^2a^{-6}$.

Next, we consider NHDE whose energy density is given by [13]

$$\rho_{\rm de} = 3(\alpha H^2 + \beta\dot{H}), \tag{48}$$

where α and β are both constants. By comparing with Eq. (30), we obtain

$$L^{-2} = \alpha H^2 + \beta\dot{H}. \tag{49}$$

By replacing it in Eq. (11) with pressureless DM component, we get the pressure corresponding to NHDE as follows:

$$p_{\rm de} = -\frac{2}{3\alpha}\rho_{\rm de} + (2\alpha - 3\beta)\frac{H^2}{\beta}. \tag{50}$$

The corresponding deceleration parameter turns out to be

$$q = \frac{1}{\beta}\left(\alpha - \beta - \frac{1}{1+r}\right). \tag{51}$$

With the help of Eqs. (50) and (30), we can rewrite Eq. (35) as $\dot{L} = \frac{HL}{2}[3 - \frac{2}{\alpha} + H^2L^2(\frac{2\alpha-3\beta}{\beta} + \kappa)]$. Consequently, we can obtain the total entropy that depends on $(H, L, \alpha, \beta, \kappa)$. Consider this equation in terms of the coincidence parameter, taking Eq. (30) into account, we obtain

$$\frac{\dot{S}_{\rm nhde}}{16\pi^2L} = 2 + \left(1 - \frac{1}{\beta} + \frac{\alpha}{\beta}H^2L^2\right)\left(1 - \frac{2}{\beta} + H^2L^2\left(\frac{2\alpha-3\beta}{\beta}\right) + \kappa\right) - \frac{1}{4}L^2\left(1 - \frac{2}{\beta} + H^2L^2\left(\frac{2\alpha-3\beta}{\beta}\right) + \kappa\right).$$

After canceling the terms and rearranging the above equation, we get

$$\frac{H\dot{S}_{\rm nhde}}{8\pi^2(1+r)} = 2 + \left(1 - \frac{1}{\beta} + \frac{\alpha}{\beta}\left(\frac{1+r}{1-\frac{b^2a^{-6}}{6H^2}}\right)\right)\left(1 - \frac{2}{\beta} + \frac{1+r}{1-\frac{b^2a^{-6}}{6H^2}}\left(\frac{2\alpha-3\beta}{\beta}\right) + \kappa\right) - \frac{1}{4}L^2\left(1 - \frac{2}{\beta} + \frac{1+r}{1-\frac{b^2a^{-6}}{6H^2}}\left(\frac{2\alpha-3\beta}{\beta}\right) + \kappa\right)b^2a^{-6}.$$

Further, we obtain

$$\dot{H} = -\frac{1}{2}\left[\frac{3(1+r)}{L^2} - \frac{2}{\beta L^2} + (2\alpha - 3\beta)\frac{H^2}{\beta} + \frac{1}{2}b^2a^{-6}\right]. \tag{52}$$

Substituting the value from Eq. (52), we get

$$\dot{L} - HL = \frac{r'H}{2L\left(H^2 - \frac{b^2a^{-6}}{6}\right)} + \frac{(1+r)H}{2L\left(H^2 - \frac{b^2a^{-6}}{6}\right)^2} \times \left(\frac{3(1+r)}{L^2} - \frac{2}{\beta L^2} + \frac{(2\alpha - 3\beta)H^2}{\beta} + \frac{1}{2}b^2a^{-6}\right) - \frac{(1+r)b^2a^{-6}H}{2L\left(H^2 - \frac{b^2a^{-6}}{6}\right)^2} - HL.$$

By considering Eqs. (52), (28) and the above equation, we get

$$\frac{\dot{S}_{\rm nhde}}{16\pi^2 L} = \left(\zeta - \frac{1}{4}L^2b^2\right)\xi + HL,$$

where $\zeta = (1 + \frac{1}{2}(\frac{3(1+r)}{L^2} - \frac{2}{\beta L^2} + \frac{(2\alpha-3\beta)H^2}{\beta} + \frac{1}{2}b^2a^{-6})L^2)$, $\xi = [\frac{r'H}{2L(H^2-\frac{b^2a^{-6}}{6})} + \frac{(1+r)H}{2L(H^2-\frac{b^2a^{-6}}{6})^2}(\frac{3(1+r)}{L^2} - \frac{2}{\beta L^2} + \frac{(2\alpha-3\beta)H^2}{\beta} + \frac{1}{2}b^2a^{-6}) - \frac{(1+r)b^2a^{-6}H}{2L(H^2-\frac{b^2a^{-6}}{6})^2} - HL]$. This expression depends on $(r,\ \beta,\ \alpha,\ b^2,\ L^2,\ a^{-6})$. It is noted that GSLT is respected for $\zeta \geq \frac{1}{4}L^2b^2$.

6. Conclusion

We have investigated GSLT by assuming the interaction of dark energy and dark matter in Chern–Simons modified gravity. We have considered a family of HDE models by assuming its various cutoffs such as Hubble horizon, event horizon, their combination, Ricci scalar and its generalized form. Firstly, we have developed the general scenario of GSLT in terms of coincidence parameter (r). Also, we have constructed the GSLT expressions for the above mentioned HDE models. The results can be summarized as follows. For HDE with Hubble horizon, GSLT is respected throughout the region. In case of event horizon, it can be noted that GSLT depends upon the cosmic coincidence parameter and Chern–Simons modified gravity parameter (b^2). However, if r behave like increasing function over late times, then r' is positive function. For this scenario, GSLT is respected along with conditions $0 < b^2 \ll 1$ and $r < -1 - 1/c^2$.

The GSLT remains valid by the evolution of the parameters in the linear combination of apparent and event horizon, i.e. $\dot{S}_{\rm ae} \geq 0$ if $\lambda_{\rm app} \geq (\frac{c^2r'}{2} + c^2(1+r) + b^2a^{-6}H + \frac{b^2}{4}L^2a^{-6})(\lambda_{\rm app} - \lambda_{\rm event}) + [\sqrt{1 - \frac{b^2a^{-6}}{6H^2}}(\lambda_{\rm app} - \lambda_{\rm event})^2 + \frac{1}{\sqrt{1-\frac{b^2a^{-6}}{6H^2}}}]c\sqrt{1+r} + \lambda_{\rm event}$ for $6H^2 \geq b^2a^{-6}$. For Ricci DE, we introduce two auxiliary parameters such as $G = 1 + \frac{3c^2(1+r)}{2} + \frac{H^2}{L^2} - 2 + \frac{1}{2}L^2b^2a^{-6}$ and $F = (\frac{c^2r'}{1-\frac{b^2a^{-6}}{6H^2}} + \frac{c^2(1+r)(\frac{3c^2(1+r)}{2L^2} + \frac{H^2}{2} - \frac{1}{L^2} + \frac{1}{4}b^2a^{-6})}{H^2(1-\frac{b^2a^{-6}}{6H^2})^2} - \frac{c^2(1+r)b^2a^{-6}}{(1-\frac{b^2a^{-6}}{6H^2})^2} - \frac{c^2(1+r)}{1-\frac{b^2a^{-6}}{6H^2}})$. It is observed that GSLT is satisfied when both parameters are positive with condition

$G + \frac{c^2(1+r)}{H(1-\frac{b^2 a^{-6}}{6H^2})} \geq \frac{1}{4}L^2$ for $6H^2 \geq b^2 a^{-6}$. In case of NHDE, it has been noted that GSLT respected for $\zeta \geq \frac{1}{4}L^2 b^2$.

The general results of GSLT have been presented in the presence of dynamical Chern–Simons modified gravity, HDE and cosmic coincidence parameters. We have also pointed out various constraints which allow the validity of GSLT. In future, we will try to elaborate the clear view of GSLT by assuming various entropy corrections model of HDE with generalized IR cutoff as mentioned in [43].

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