Thailand Statistician
April 2022; 20(2): 372-394
http//:statassoc.or.th
Contributed paper

Characterization of the New Modified Class of Power Function Distribution with Theory, Simulation and Applications

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Received: 27 March 2020
Revised: 8 September 2020
Accepted: 13 November 2020

Abstract

The generalization of the probability distribution has gained great attention in statistical field. In this study, a new Kumaraswamy Lehmann-2 power function distribution (KL2PFD) is proposed. We suggest a new generator that will modify the power function distribution called Kumaraswamy Lehmann-2 generator (KL2-G). The various properties of the new distribution have been discussed in detail such as moments, vitality function, conditional moments and order statistics etc. We have also characterized the KL2PFD based on, conditional moments (Right and Left Truncated mean) and doubly truncated mean. The shape of the new distribution has been studied for applied sciences. The aim of the study is to increase the application of the Power function distribution. This distribution can be used for approximately symmetric data (normal data), positive and negative skewed data. For this, we have studied the real life application of the distribution by using four different data sets. After analyzing data, we conclude that the proposed model KL2PFD perform better in all the data sets while compared to different competitor models. It is hoped that the findings of this paper will be useful for researchers in different field of applied sciences.

Keywords: Characterization of truncated distribution, entropies, Kumaraswamy Lehmann-2 power function distribution, Lehmann alternatives, power function distribution.

1. Introduction

The researchers in engineering sciences mostly study the reliability of different components by taking the help from probability distributions that are simple in mathematical expression instead of using mathematically complex probability distributions. Dallas (1976) introduced the power function as the inverse of Pareto distribution. Meniconi and Barry (1996) showed that power function distribution is better to fit for failure data over exponential, lognormal and Weibull because it provides a better fit.

More studies about the application of this distribution and its applications can be found in Ahsanullah (2013), Dorp et al. (2002) and Chang (2007). For modeling heterogeneous population, Saleem et al. (2010) talked about the two component mixture of one-parameter power function distribution. Estimation of the parameters of the two-parameter power function distribution was studied by Zaka and Akhter (2013) through the methods of least squares, relative least squares and ridge regression. According to its applicability in real life situations for modeling survival data, Tahir et al. (2014) proposed the modification of the power function distribution as Weibull-power function distribution. By using the Bayesian inference, Hanif, et al. (2015) estimated the parameter of the one-parameter power function distribution. Shahzad and Asghar (2016) introduced the transmuted power function distribution by following Shaw and Buckley (2009). Okorie et al. (2017) proposed the modification of the power function distribution by using Marshall and Olkin (1997) technique. Haq et al. (2018) introduced the McDonald power function distribution. Jabeen and Zaka (2019) discussed the parameters estimation for continuous uniform distribution using modified percentile estimators. Zaka et al. (2020) proposed the reflected and exponentiated class of power function distribution. Zaka et al. (2020) introduced the beta Lehmann 2 power function distribution.

The cumulative distribution function (cdf) and probability density function (pdf) of the power function distribution is given below

$$G(x)=\left(\frac{x}{\beta}\right)^{\gamma}, \tag{1}$$

$$g(x)=\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}, \tag{2}$$

Lehmann alternatives were introduced by Lehmann (1953) in the two-sample hypothesis testing context and are useful in survival analysis. The cumulative density function for Lehmann 2 relationship is given as

$$F(x)=1-\{1-G(x)\}^{\theta}\ . \quad \text{(Lehmann 2 relationship)}$$

Cordeiro et al. (2011) introduced the Kumaraswamy generator. Then, the mixture of these two techniques is known as Kumaraswamy Lehmann-2 generator (KL2-G). The cdf and pdf of the Kw-Leh2-generator are given as

$$F(x)=1-\left\{1-\left\{1-\{1-G(x)\}^{\theta}\right\}^{\alpha}\right\}^{\phi}, \tag{3}$$

and

$$f(x)=\alpha\phi\theta\left\{1-\{1-G(x)\}^{\theta}\right\}^{\alpha-1}\{1-G(x)\}^{\theta-1}\left\{1-\left\{1-\{1-G(x)\}^{\theta}\right\}^{\alpha}\right\}^{\phi-1}g(x), \tag{4}$$

where $G(x)$ is the cdf and $g(x)$ is the pdf of any probability distribution.

In this paper, we suggest a new distribution that will generalize the power function distribution (PFD) by using the above mentioned technique. We have derived some of the main structural properties of this distribution. The application of this distribution is illustrated by an application to real life data sets. It is hoped that the findings of this study will be useful for researchers in different field of applied sciences.

1.1. Model identification for KL2PFD

The pdf and cdf of power function distribution are given as follows:

$$g(x)=\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}};0<x<\beta,\gamma>0, \tag{5}$$

and
$$G(x)=\left(\frac{x}{\beta}\right)^{\gamma}, \tag{6}$$

where γ and β are the shape and scale parameters. Following the generator (3), the KL2PFD is obtained by putting (5) and (6) in (4) and simplifying, we get

$$f(x)=\alpha\phi\theta\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha-1}\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta-1}\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi-1}\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}};0<x<\beta, \tag{7}$$

and associated cdf is obtained by putting (5) and (6) in (3) as

$$F(x)=1-\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}. \tag{8}$$

We may observe α,β and γ as the shape and β as scale parameters.

1.2. Expansion of cdf

Considered the expansion of power series for any real non-integer as follows

$$(1-z)^{b-1}=\Sigma\frac{(-1)^{k}\Gamma(b)}{\Gamma(b-k)k!}. \tag{9}$$

We may see that for $|z|<1,$ the expression in (9) may be used. The cdf of the distribution may be written as by using the binomial expansion as

$$F(x)=1-\sum_{k=0}^{\infty}w_k H_k(x),$$

where

$$\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(\phi+1)}{\Gamma(\phi+1-j)j!}\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha j},$$

$$\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(\phi+1)\Gamma(\alpha j+1)}{\Gamma(\phi+1-j)j!\Gamma(\alpha j+1-i)i!}\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta i},$$

$$\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\phi+1)\Gamma(\alpha j+1)\Gamma(\theta i+1)}{\Gamma(\phi+1-j)j!\Gamma(\alpha j+1-i)i!\Gamma(\theta i+1-k)k!}\left(\frac{x}{\beta}\right)^{\gamma k},$$

$$w_k=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\phi+1)\Gamma(\alpha j+1)\Gamma(\theta i+1)}{\Gamma(\phi+1-j)j!\Gamma(\alpha j+1-i)i!\Gamma(\theta i+1-k)k!},$$

and

$$H_k(x)=(G(x))^{k}=\left(\left(\frac{x}{\beta}\right)^{\gamma}\right)^{k}.$$

Also, $f(x)=\sum_{l=0}^{\infty} t_l h_{l+1}(x)$ where

$$t_l=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{k+i+j+l}\,\alpha\phi\gamma(l+1)\Gamma(\alpha)\Gamma(\phi)\Gamma(\alpha i+1)\Gamma(\theta(j+k+1))}{\Gamma(\alpha-j)\Gamma(\phi-i)\Gamma(\alpha i+1-k)\Gamma(\theta(j+k+1))\,j!k!i!(l+1)},$$

and

$$h_{l+1}(x)=\frac{\gamma(l+1)x^{\gamma(l+1)-1}}{\beta^{\gamma(l+1)}}.$$

By definition, the survival function is $S(x)=1-F(x)$. We know that $F(x)=1-\sum_{k=0}^{\infty} w_k H_k(x)$. So, $S(x)=\sum_{k=0}^{\infty} w_k H_k(x)$. By definition, the hazard rate function (HRF) of probability distribution is given as $H(x)=\frac{f(x)}{S(x)}$. That may be generated for KL2PFD as $H(x)=\frac{\sum_{l=0}^{\infty} t_l h_{l+1}(x)}{\sum_{k=0}^{\infty} w_k H_k(x)}$.

1.3. Shapes of KL2PFD

The KL2PFD can be approximately normal curve, whereas the HRF can be bathtub, monotonically increasing and decreasing shapes. (See Figures 1-3).

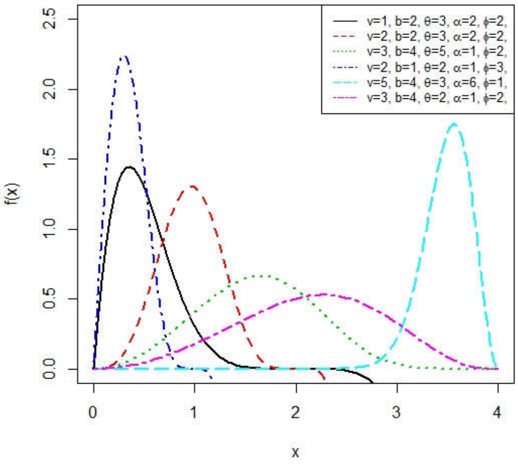

Figure 1 Plots of PDF of KL2PFD

2. Some Basic Properties of KL2PFD

2.1. Moments about zero

The r^{th} moments about zero of any distribution is described below

$$\mu_r'=\int_0^{\beta} x^r f(x)\,dx.$$

By solving we get

$$\mu_r' = \frac{\phi\, a_j\, a_i\, a_k\, \beta^r}{j + \dfrac{k}{\alpha} + 1},$$

where $a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\phi)}{\Gamma(\phi - j)\, j!}, a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma\left(\dfrac{r}{\gamma}+1\right)}{\Gamma\left(\dfrac{r}{\gamma}+1-i\right) i!}$ and $a_k = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\dfrac{i}{\theta}+1\right)}{\Gamma\left(\dfrac{i}{\theta}+1-k\right) k!}.$

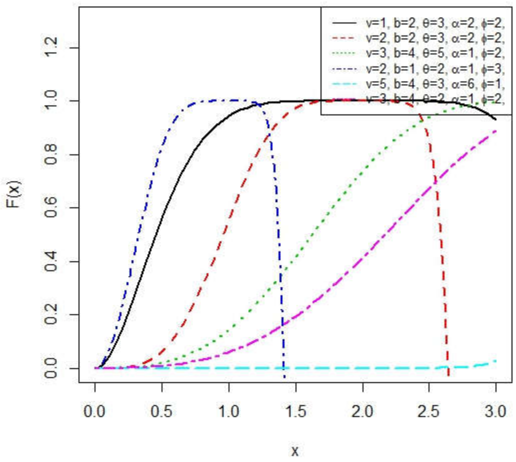

Figure 2 Plots of cdf of KL2PFD

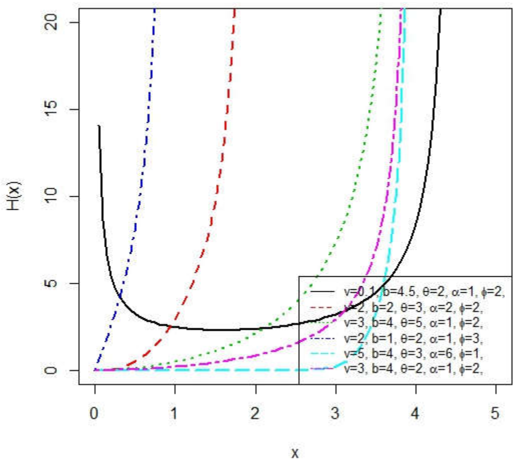

Figure 3 Plots of HRF of KL2PFD

2.2. Moment generating functions

Apart from generating functions, the moment generating function can be utilized to describe the characteristic of the random variable. The moment generating function may be defined as the linear combination of exponential generalized univariate distributions as $M_0(t) = \int_0^{\beta} e^{tx} \sum_{l=0}^{\infty} t_l h_{l+1}(x).$

If X follows KL2PFD, the moment generating function may be derived as

$$M_0(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\phi a_j a_i a_k \beta^r}{j + \frac{k}{\alpha} + 1},$$

where $a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\phi)}{\Gamma(\phi - j) j!}$, $a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma\left(\frac{r}{\gamma} + 1\right)}{\Gamma\left(\frac{r}{\gamma} + 1 - i\right) i!}$ and $a_k = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{i}{\theta} + 1\right)}{\Gamma\left(\frac{i}{\theta} + 1 - k\right) k!}$.

2.3. Random number generator

The random number may be obtained from

$$\mathrm{R} = 1 - \left\{1 - \left\{1 - \left\{1 - \left(\frac{\mathrm{x}}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\varphi}.$$

After simplifying, we get

$$x = \left[1 - \left\{1 - \left\{1 - (1 - R)^{1/\phi}\right\}^{1/\alpha}\right\}^{\frac{1}{\theta}}\right]^{\frac{1}{\gamma}},$$

where $R = F(x)$.

3. Inverse Moments

By definition Inverse moments may be obtained as $\mu'_{-r} = \int_0^{\beta} x^{-r} \sum_{l=0}^{\infty} t_l h_{l+1}(x) dx.$ We get inverse moments for KL2PFD as

$$\mu'_{-r} = \frac{\phi a_j a_i a_k \beta^{-r}}{j + \frac{k}{\alpha} + 1},$$

where $a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\phi)}{\Gamma(\phi - j) j!}, a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma\left(\frac{-r}{\gamma} + 1\right)}{\Gamma\left(\frac{-r}{\gamma} + 1 - i\right) i!}$ and $a_k = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{i}{\theta} + 1\right)}{\Gamma\left(\frac{i}{\theta} + 1 - k\right) k!}$.

3.1. Mean residual function

The mean residual function tells us the time to be expected for survival of an individual provided that one already has reached this time point $e(x) = \int_x^{\beta} \frac{S(t)}{S(x)} dt.$ For Kumaraswamy Lehmann-2 power function distribution (KL2PFD), we get mean residual function as

$$e(x)=\frac{\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{\beta^{\gamma k+1}-x^{\gamma k+1}}{\gamma k+1}\right\}}{\sum_{k=0}^{\infty}w_k H_k(x)},$$

where $a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\phi+1)}{\Gamma(\phi+1-j)\,j!}, a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(\alpha j+1)}{\Gamma(\alpha j+1-i)\,i!}$ and $a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\theta i+1)}{\Gamma(\theta i+1-k)\,k!}.$

3.2. Vitality function

The vitality function is obtained for KL2PFD as $V(x)=\frac{1}{S(x)}\int_x^{\beta}x\,f(x)\,d(x)$. That may be obtained as

$$V(x)=\frac{\dfrac{\beta\phi\,a_j\,a_i\,a_k\left[1-\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\left(j+\frac{k}{\alpha}+1\right)}\right]}{\left(j+\frac{k}{\alpha}+1\right)}}{\sum_{k=0}^{\infty}w_k H_k(x)},$$

where $a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\phi)}{\Gamma(\phi-j)\,j!}, a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma\left(\frac{1}{\gamma}+1\right)}{\Gamma\left(\frac{1}{\gamma}+1-i\right)i!}$ and $a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma\left(\frac{i}{\theta}+1\right)}{\Gamma\left(\frac{i}{\theta}+1-k\right)k!}.$

3.3. Information function

The information function is given as $IF=\int_0^{\beta}\left\{f(x)\right\}^s dx.$ For KL2PFD the information function is given as

$$IF=\frac{\dfrac{\alpha^{s-1}\phi^s(\theta\gamma)^{s-1}}{\beta^{\gamma(s-1)}}a_j\,a_i\,a_k\,a_l}{\beta^{(s-1)(\gamma-1)}\left(\dfrac{(s-1)(\alpha-1)(j+1)}{\alpha}+i+1\right)},$$

where $a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma\left(\frac{(s-1)(\theta-1)}{\alpha}+1\right)}{\Gamma\left(\frac{(s-1)(\theta-1)}{\alpha}+1-j\right)j!}, a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(s(\phi-1)+1)}{\Gamma(s(\phi-1)+1-i)\,i!},$

$a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma\left(\frac{(s-1)(\gamma-1)}{\gamma}+1\right)}{\Gamma\left(\frac{(s-1)(\gamma-1)}{\gamma}+1-k\right)k!}$ and $a_l=\sum_{l=0}^{\infty}\frac{(-1)^l\,\Gamma\left(\frac{k}{\theta}+1\right)}{\Gamma\left(\frac{k}{\theta}+1-l\right)l!}.$

4. Reverse Hazard Function

The reverse hazard function may be written as $r(x)=\frac{f(x)}{F(x)}$ that may be simplify for KL2PFD as $r(x)=\frac{\sum_{l=0}^{\infty} t_l h_{l+1}(x)}{1-\sum_{k=0}^{\infty} w_k H_k(x)}$.

4.1. Mills ratio

We may write mills ratio as $m(x)=\frac{S(x)}{f(x)}=\frac{\sum_{k=0}^{\infty} w_k H_k(x)}{\sum_{l=0}^{\infty} t_l h_{l+1}(x)}$.

4.2. Order statistics

The pdf of the order statistic may be written as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\left(F(x)\right)^{i}\left\{1-F(x)\right\}^{n-i} f(x).$$

For KL2PFD, we may write the order statistics as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\left(\sum_{k=0}^{\infty} w_k H_k(x)\right)^{i}\left\{1-\sum_{k=0}^{\infty} w_k H_k(x)\right\}^{n-i}\left(\sum_{l=0}^{\infty} t_l h_{l+1}(x)\right).$$

4.3. Shanon entropy and Renyi entropy

The Shanon entropy may be defined as $H(x)=-E\left\{\log f(x)\right\}$,

$$H(x)=-\left[\log\left(\frac{\alpha\phi\theta^{\gamma}}{\beta^{\gamma}}\right)+\left\{\begin{array}{c}(\gamma-1)\left\{a_j\, a_i\, a_k\,\frac{\beta^{2\gamma}}{\gamma(k+1)}\right\}+(\alpha-1)\alpha\varphi a_j\left\{\frac{-1}{\left(\alpha(j+1)\right)^2}\right\}+\\ (\theta-1)\alpha\varphi a_j\, a_i\, a_k\left\{\frac{-1}{\left(\theta(j+k+1)\right)^2}\right\}+\frac{(1-\phi)}{\phi}\end{array}\right\}\right],$$

and Renyi entropy $H_r(x)=\frac{1}{1-r}\log\left(\int_0^{\beta}\left(f(x)\right)^{r}\right)dx$,

$$H_r(x)=\frac{1}{1-r}\left\{\frac{\frac{\alpha^{r-1}\phi^{r}(\theta\gamma)^{r-1}}{\beta^{\gamma(r-1)}}a_j\, a_i\, a_k\, a_l}{\beta^{(r-1)(\gamma-1)}\left(\frac{(r-1)(\alpha-1)(j+1)}{\alpha}+i+1\right)}\right\},$$

where $a_j=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma\left(\frac{(r-1)(\theta-1)}{\alpha}+1\right)}{\Gamma\left(\frac{(r-1)(\theta-1)}{\alpha}+1-j\right)j!}, a_i=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma\left(r(\phi-1)+1\right)}{\Gamma\left(r(\phi-1)+1-i\right)i!}$,

$$a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma\left(\frac{(r-1)(\gamma-1)}{\gamma}+1\right)}{\Gamma\left(\frac{(r-1)(\gamma-1)}{\gamma}+1-k\right)k!}\text{ and } a_l=\sum_{l=0}^{\infty}\frac{(-1)^l\,\Gamma\left(\frac{k}{\theta}+1\right)}{\Gamma\left(\frac{k}{\theta}+1-l\right)l!}.$$

5. Incomplete Moments

The incomplete moments are given as $\mu_{X|(\alpha,\beta,\gamma,\phi,\theta);r}(p)=\int_0^P x^r f(x)\,dx.$ By simplifying for KL2PFD, we get

$$\mu_{X|(\alpha,\beta,\gamma,\phi,\theta);r}(p)=\frac{\phi\, a_j\, a_i\, a_k\,\beta^r\left[1-\left\{1-\left(\frac{p}{\beta}\right)^{\gamma}\right\}^{\theta}\right]^{\alpha\left(j+\frac{k}{\alpha}+1\right)}}{j+\frac{k}{\alpha}+1},$$

where $a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\phi)}{\Gamma(\phi-j)\,j!}$, $a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma\left(\frac{r}{\gamma}+1\right)}{\Gamma\left(\frac{r}{\gamma}+1-i\right)i!}$ and $a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma\left(\frac{i}{\theta}+1\right)}{\Gamma\left(\frac{i}{\theta}+1-k\right)k!}.$

5.1. Conditional moments

The conditional moments may be obtained as $E\left[X^r\mid X>t\right]=\frac{1}{\bar{F}(t)}\int_t^{\beta}x^r\sum_{l=0}^{\infty}t_j h_{l+1}(x)\,dx.$ The conditional moments for KL2PFD may be obtained by using above expression as

$$E\left[X^r\mid X>t\right]=\frac{1}{\bar{F}(t)}\frac{\phi\, a_j\, a_i\, a_k\,\beta^r\left[1-\left[1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right]^{\alpha\left(j+\frac{k}{\alpha}+1\right)}\right]}{j+\frac{k}{\alpha}+1},$$

where $a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\phi)}{\Gamma(\phi-j)\,j!}$, $a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma\left(\frac{r}{\gamma}+1\right)}{\Gamma\left(\frac{r}{\gamma}+1-i\right)i!}$ and $a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma\left(\frac{i}{\theta}+1\right)}{\Gamma\left(\frac{i}{\theta}+1-k\right)k!}.$

5.2. Lorenz and Bonferroni curve

The Lorenz and Bonferroni curve may be obtained as

$$L(p)=\frac{1}{\mu}\int_0^q x\sum_{l=0}^{\infty}t_l h_{l+1}(x)\,dx=\frac{1}{\mu}\frac{\phi\, a_j\, a_i\, a_k\,\beta\left[\left[1-\left\{1-\left(\frac{q}{\beta}\right)^{\gamma}\right\}^{\theta}\right]^{\alpha\left(j+\frac{k}{\alpha}+1\right)}\right]}{j+\frac{k}{\alpha}+1},$$

$$B(p)=\frac{1}{P\mu}\frac{\phi a_j\, a_i\, a_k\, \beta\left[\left[1-\left\{1-\left(\frac{q}{\beta}\right)^{\gamma}\right\}^{\theta}\right]^{\alpha\left(j+\frac{k}{\alpha}+1\right)}\right]}{j+\frac{k}{\alpha}+1},$$

where $a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\phi)}{\Gamma(\phi-j)\,j!}$, $a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma\left(\frac{r}{\gamma}+1\right)}{\Gamma\left(\frac{r}{\gamma}+1-i\right)i!}$ and $a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma\left(\frac{i}{\theta}+1\right)}{\Gamma\left(\frac{i}{\theta}+1-k\right)k!}$.

6. Characterization of KL2PFD

Let X be KL2PF variable with probability density function

$$f(x)=\alpha\phi\theta\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha-1}\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta-1}\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi-1}\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}};0<x<\beta,$$

and let $\bar{F}(x)$ be the survival function respectively. Then, the random variable X has KL2PFD if and only if

$$V(X\mid x\le t)=\frac{1}{F(t)}\left[(-1)t^2\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+2\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{2+\gamma k}}{2+\gamma k}\right\}\right]$$
$$-\left[\frac{1}{F(t)}\left[(-1)t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{1+\gamma k}}{1+\gamma k}\right\}\right]\right]^2,$$

where $V(X\mid x\le t)$ is conditional variance $a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\phi+1)}{\Gamma(\phi+1-j)\,j!}$, $a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(\alpha j+1)}{\Gamma(\alpha j+1-i)\,i!}$,

and $a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\theta i+1)}{\Gamma(\theta i+1-k)\,k!}$.

Proof: Necessary part:

$$E(X^r\mid x\le t)=\frac{1}{F(t)}\int_0^t x^r\alpha\phi\theta\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha-1}\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta-1}\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi-1}\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}dx$$

$$=\frac{1}{F(t)}\left[(-1)x^r\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}\Bigg|_0^t-\int_0^t(-1)rx^{r-1}\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}dx\right]$$

$$=\frac{1}{F(t)}\left[(-)t^{r}\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+r\int_{0}^{t}x^{r-1}\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}dx\right],$$

where

$$\int_{0}^{t}x^{r-1}\left\{1-\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}dx=\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{r+\gamma k}}{r+\gamma k}\right\},$$

and

$$a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\phi+1)}{\Gamma(\phi+1-j)\,j!},\ a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(\alpha j+1)}{\Gamma(\alpha j+1-i)\,i!},\ \text{and}\ a_k=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\theta i+1)}{\Gamma(\theta i+1-k)\,k!}.$$

Therefore, $E\left(X^{r}\mid x\le t\right)$ becomes

$$E\left(X^{r}\mid x\le t\right)=\frac{1}{F(t)}\left[(-1)t^{r}\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+r\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{r+\gamma k}}{r+\gamma k}\right\}\right].$$

Put $r=1$,

$$E\left(X\mid x\le t\right)=\frac{1}{F(t)}\left[(-1)t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{1+\gamma k}}{1+\gamma k}\right\}\right].$$

Put $r=2$,

$$E\left(X^{2}\mid x\le t\right)=\frac{1}{F(t)}\left[(-1)t^{2}\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+2\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{2+\gamma k}}{2+\gamma k}\right\}\right]$$

$$\begin{aligned}V\left(X\mid x\le t\right)=&\frac{1}{F(t)}\left[(-1)t^{2}\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+2\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{2+\gamma k}}{2+\gamma k}\right\}\right]\\&-\left[\frac{1}{F(t)}\left[(-1)t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{1+\gamma k}}{1+\gamma k}\right\}\right]\right]^{2}\end{aligned}\tag{10}$$

Also, sufficient part

$$V\left(X\mid x\le t\right)=\frac{1}{F(t)}\int_{0}^{t}x^{2}dx-\left\{\frac{1}{F(t)}\int_{0}^{t}xdx\right\}^{2}=t^{2}-2\int_{0}^{t}\frac{xF(x)}{F(t)}dx-\left\{t-\int_{0}^{t}\frac{F(x)}{F(t)}dx\right\}^{2}.\tag{11}$$

Equate (10) and (11), we get

$$t^2-2\int_0^t \frac{xF(x)}{F(t)}dx-\left\{t-\int_0^t \frac{F(x)}{F(t)}dx\right\}^2=\frac{1}{F(t)}\left[(-1)t^2\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+2\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{2+\gamma k}}{2+\gamma k}\right\}\right]$$

$$-\left[\frac{1}{F(t)}\left[(-1)t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{1+\gamma k}}{1+\gamma k}\right\}\right]\right]^2.$$

As $t-\int_0^t \frac{F(x)}{F(t)}dx=\frac{1}{F(t)}\left[(-1)t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{1+\gamma k}}{1+\gamma k}\right\}\right]$, therefore

$$t^2-2\int_0^t \frac{xF(x)}{F(t)}dx=\frac{1}{F(t)}\left[(-1)t^2\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+2\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}\left\{\frac{t^{2+\gamma k}}{2+\gamma k}\right\}\right].$$

Differentiate w.r.t. "t",

$$t^2f(t)=t^2\,\alpha\phi\theta\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha-1}\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta-1}\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi-1}\frac{\gamma t^{\gamma-1}}{\beta^{\gamma}}$$

$$-2t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+2\frac{a_j\,a_i\,a_k}{\beta^{\gamma k}}t^{\gamma k+1}.$$

Also, $a_ja_ia_k\left\{\frac{t^{\gamma k}}{\beta^{\gamma k}}\right\}=\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}$,

$$t^2f(t)=t^2\,\alpha\phi\theta\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha-1}\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta-1}\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi-1}\frac{\gamma t^{\gamma-1}}{\beta^{\gamma}}$$

$$-2t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi}+2t\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi},$$

$$f(t)=\alpha\phi\theta\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha-1}\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta-1}\left\{1-\left\{1-\left\{1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\theta}\right\}^{\alpha}\right\}^{\phi-1}\frac{\gamma t^{\gamma-1}}{\beta^{\gamma}}.$$

7. Comparison between Maximum Likelihood and Percentile Estimation Methods of the Parameters of KL2PFD

7.1. Maximum likelihood method (MLM)

Let $x_1,...,x_n$ be a random sample of size n from the KL2PFD. The log-likelihood function for the KL2PFD is given by

$$L(\gamma,\beta,\alpha,\phi,\theta)=n\ln\left(\frac{\alpha\phi\theta\gamma}{\beta^{\gamma}}\right)+(\alpha-1)\sum_{i=1}^{n}\ln\left\{1-\left(1-\left(\frac{x_i}{\beta}\right)^{\gamma}\right)^{\theta}\right\}+(\theta-1)\ln\left(1-\left(\frac{x_i}{\beta}\right)^{\gamma}\right)$$

$$+(\gamma-1)\ln x_i+(\phi-1)\ln\left[1-\left\{1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\theta}\right\}^{\alpha}\right].$$

The score vector are

$$U_{\beta}(\gamma,\beta,\alpha,\phi,\theta)=\frac{\partial}{\partial\beta}L(\gamma,\beta,\alpha,\phi,\theta),\quad U_{\gamma}(\gamma,\beta,\alpha,\phi,\theta)=\frac{\partial}{\partial\gamma}L(\gamma,\beta,\alpha,\phi,\theta),$$

$$U_{\alpha}(\gamma,\beta,\alpha,\phi,\theta)=\frac{\partial}{\partial\alpha}L(\gamma,\beta,\alpha,\phi,\theta),\quad U_{\phi}(\gamma,\beta,\alpha,\phi,\theta)=\frac{\partial}{\partial\phi}L(\gamma,\beta,\alpha,\phi,\theta),$$

$$U_{\theta}(\gamma,\beta,\alpha,\phi,\theta)=\frac{\partial}{\partial\theta}L(\gamma,\beta,\alpha,\phi,\theta).$$

The parameters of KL2PFD can be obtained by solving the above equations resulting from setting the five partial derivatives of $L(\gamma,\beta,\alpha,\phi,\theta)$ equals to zero.

7.2. Estimation of KL2PFD parameters from common percentiles

Dubey (1967) proposed a percentile estimator of the shape parameter, based on any two sample percentiles. After Marks (2005) also discussed it, in which he estimated the parameters of Weibull distribution using percentiles. Marks (2005) called it common percentile method.

Let $x_1,...,x_n$ be a random sample of size n drawn from the probability density function of KL2PFD. The cumulative distribution function of a KL2PFD with shape and scale parameters γ and β, respectively is

$$F(x)=1-\left[1-\left\{1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\theta}\right\}^{\alpha}\right]^{\phi}.$$

By solving, we get

$$x=\left[1-\left\{1-\left(1-(R)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]^{1/\gamma}, \tag{12}$$

where $R=F(x)$. Let P_{75} and P_{25} are the 75th and 25th percentiles, therefore (12) becomes

$$P_{75}=\beta\left[1-\left\{1-\left(1-(0.75)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]^{1/\gamma}, \tag{13}$$

$$P_{25} = \beta\left[1-\left\{1-\left(1-\left(0.25\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]^{1/\gamma}. \tag{14}$$

Solving the above equations, we get

$$\left(\frac{P_{75}}{P_{25}}\right)^{\gamma} = \left\{\frac{\left[1-\left\{1-\left(1-\left(0.75\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}{\left[1-\left\{1-\left(1-\left(0.25\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}\right\}, \quad \gamma \ln\left(\frac{P_{75}}{P_{25}}\right) = \ln\left\{\frac{\left[1-\left\{1-\left(1-\left(0.75\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}{\left[1-\left\{1-\left(1-\left(0.25\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}\right\},$$

$$\hat{\gamma} = \frac{\ln\left\{\frac{\left[1-\left\{1-\left(1-\left(0.75\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}{\left[1-\left\{1-\left(1-\left(0.25\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}\right\}}{\ln\left(\frac{P_{75}}{P_{25}}\right)}, \text{ and } \hat{\beta} = \frac{\left(P_{75}\right)}{\left[1-\left\{1-\left(1-\left(0.75\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]^{1/\hat{\gamma}}},$$

generally

$$\hat{\gamma} = \frac{\ln\left\{\frac{\left[1-\left\{1-\left(1-\left(H\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}{\left[1-\left\{1-\left(1-\left(L\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]}\right\}}{\ln\left(\frac{P_{H}}{P_{L}}\right)}, \text{ and } \hat{\beta} = \frac{P_{H}}{\left[1-\left\{1-\left(1-\left(H\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]^{1/\hat{\gamma}}},$$

where H is the maximum percentage, L is the minimum percentage and P is the percentile.

A simulation study is used in order to compare the performance of the proposed estimation methods. We carry out this comparison taking the samples of sizes as n = 40 and 150 with pairs of (β, γ) = {(1, 2), (2, 1) and (1.5, 1.5)}. We generated random samples of different sizes by observing that if R_i is random number taking (0,1), then $x_i = \left[1-\left\{1-\left(1-\left(R\right)^{1/\phi}\right)^{1/\alpha}\right\}^{1/\theta}\right]^{1/\gamma}$ is the random number generation from KL2PFD with $(\gamma, \beta, \theta, \alpha \text{ and } \phi)$ parameters. All results are based on 5,000 replications.

Such generated data have been used to obtain estimates of the unknown parameters. The results obtained from parameters estimation of KL2PFD using different sample sizes and different values of parameters with mean square error MSE.

$$MSE\left(\hat{\beta}\right) = E\left[\left(\hat{\beta}-\beta\right)^2\right] \text{ and } MSE\left(\hat{\gamma}\right) = E\left[\left(\hat{\gamma}-\gamma\right)^2\right].$$

Table 1 Estimates for the parameters of Kw-Leh2 power function distribution with different estimation methods under the sample size 40 when $\theta = 1$, $\alpha = 2$ and $\varphi = 3$

Methods	True Values		Estimated Values		MSE	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	0.9392683	2.6823127	0.0228598	0.3413591
	2	1	2.0233855	0.9402877	0.1368475	0.1182024
	1.5	1.5	1.5500040	1.2956750	0.0796184	0.1662227
PE	1	2	0.9876744	2.1523000	0.0046620	0.2404506
	2	1	1.9638820	1.0783290	0.0723532	0.0594456
	1.5	1.5	1.4803110	1.6134530	0.0185282	0.1367111

Table 2 Estimates for the parameters of Kw-Leh2 power function distribution with different estimation methods under the sample size 150 when $\theta = 1$, $\alpha = 2$ and $\varphi = 3$

Methods	True Values		Estimated Values		MSE	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	1.0000880	2.0857430	0.0148897	0.13281223
	2	1	2.0655667	0.9422703	0.0703308	0.06061753
	1.5	1.5	1.4887550	1.5887790	0.0309661	0.10352545
PE	1	2	0.9968213	2.0430360	0.0013130	0.05096839
	2	1	1.9924040	1.0188100	0.0211052	0.01256001
	1.5	1.5	1.4933580	1.5292180	0.0052700	0.02862667

Table 3 Estimates for the parameters of Kw-Leh2 power function distribution with different estimation methods under the sample size 40 when and $\theta = 3$, $\alpha = 2$ and $\varphi = 1$

Methods	True Values		Estimated Values		MSE	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	0.9753603	2.4580265	0.0327993	0.2748554
	2	1	2.1779746	0.9633238	0.2371783	0.1196290
	1.5	1.5	1.5324380	1.4151410	0.0991182	0.1647704
PE	1	2	0.9834422	2.162290	0.0090098	0.2321259
	2	1	1.9624570	1.0742530	0.1437567	0.0547915
	1.5	1.5	1.4698180	1.6188390	0.0361879	0.1287172

Table 4 Estimates for the parameters of Kw-Leh2 power function distribution with different estimation methods under the sample size 150 when $\theta = 3$, $\alpha = 2$ and $\varphi = 1$

Methods	True Values		Estimated Values		MSE	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	1.0402820	1.9212690	0.0272471	0.1188129
	2	1	1.9357860	1.1052690	0.0910156	0.0696257
	1.5	1.5	1.5384420	1.4183290	0.0524595	0.0863524
PE	1	2	0.9972467	2.0340660	0.0025526	0.0466708
	2	1	1.9902580	1.0195430	0.0402431	0.0117741
	1.5	1.5	1.4935130	1.5276010	0.0103143	0.0258056

8. Applications

In this section, we have analyzed four real life data sets to demonstrate the performance of KL2PFD. The comparison of the probability distributions has been made in all the data sets on the basis of Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). Finally, using the above mentioned criteria's, our proposed KL2PFD is better than the different competitor models for the same data sets.

Data Set 1

We have adopted the data set given by Bader and Priest (1982) to demonstrate the performance of our proposed model. The data shows the measure of strength measured for single carbon fibers and soak at gauge lengths of 1.0, 10.0, 20.0 and 50.0 millimeter. The soaked tows of 100 fibers were tested at gauge lengths (in mm) of 20.0, 50.0, 150.0 and 300.0 mm. Here, we consider that the data set of single fibers of 20 mm in gauge with a sample of size 63. The data are: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

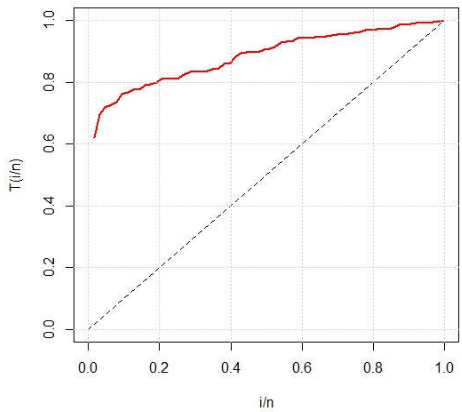

Figure 4 TTT plot for carbon fibers data

The TTT plot is displayed in Figure 4, which indicates that the HRF associated with the data set has an increasing shape, since the plot shows a first concave curvature. So, we can easily fit KL2PFD on the carbon fibers data. We have compared KL2PFD with the Kumarswamy Marshall-Olkin family of distribution (Kw-MO) (see Morad et al. (2015)) and other comparative models: the Beta-Frechet (BFr) (Barreto et al. 2011), the exponentiated Fréchet (EFr) (Nadarajah and Kotz 2003), the Marshall-Olkin extended Fréchet (MO-Fr) (Krishna and Ristic 2013). The proposed model KL2PFD is showing better results as compare to the other competitive models by providing smallest AIC, BIC, CAIC and HQIC for the given data.

Table 5 Statistics of carbon fibers data

Models	AIC	BIC	CAIC	HQIC
KL2PFD	**110.8663**	**117.2477**	**111.2801**	**113.3718**
KwMO-Fr	121.867	132.583	122.920	126.082
BFr	120.594	129.166	121.283	123.965
EFr	118.700	125.130	119.107	121.229
MO-Fr	119.746	126.175	120.153	122.275
Fr	121.804	126.091	122.004	123.490

From Table 5, it is clear that the KL2PFD provides better fit for the above data set as it provides minimum AIC, BIC, CAIC and HQIC.

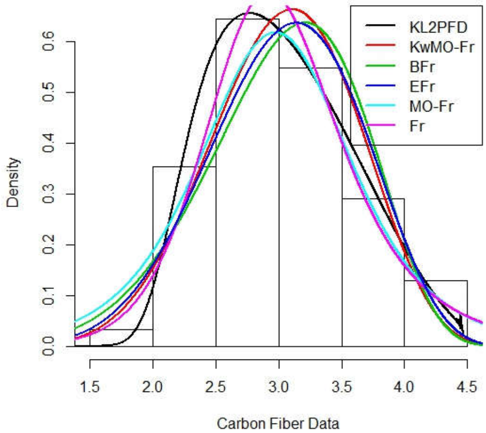

Figure 5 Estimated density plot for carbon fibers data

Data Set 2

We have adopted the data set consisting the remission time of 128 bladder cancer patients to demonstrate the performance of our proposed KL2PFD. These data were also studied by Tahir et al. (2014), Lemonte (2014), Zea et al. (2012), Lee and Wang (2003) and Lemonte and Cordeiro (2013). The remission times in month are given below: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

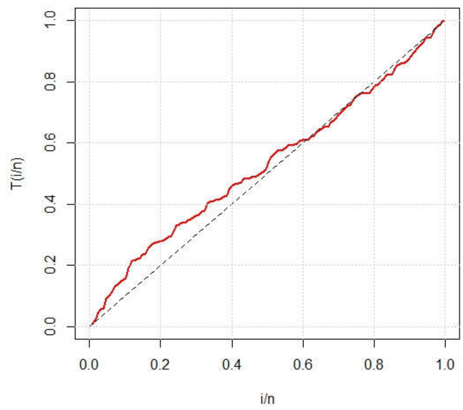

Figure 6 TTT Plot for bladder cancer data

The TTT plot of the remission time (in month) for bladder cancer patients is exhibited in Figure 6, we may see that the hazard rate function has little bit bathtub shape, so we may easily fit KL2PFD on the bladder cancer data.

Table 6 Statistics of Bladder Cancer Data

Models	AIC	BIC	CAIC	HQIC
KL2PFD	**810.1383**	**810.4661**	**821.515**	**814.7605**
WP	818.9331	819.0298	824.6214	821.2442
KwP	824.4200	832.9525	824.6151	827.8866
BEP	826.1318	837.5085	826.4596	830.7540
BP	843.8620	852.3946	844.0571	847.3287
Pareto	1081.1820	1084.0260	1081.2140	1082.3380

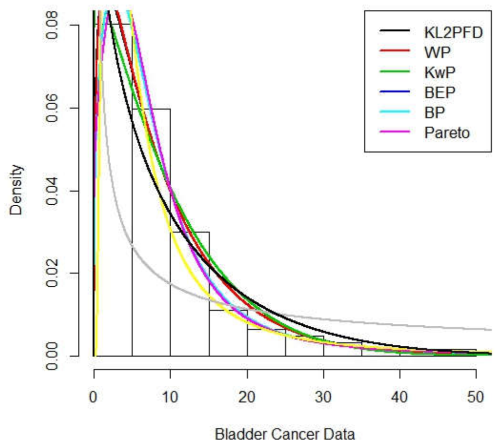

Figure 7 Estimated density plot for bladder cancer data

From Table 6, we may see that KL2PFD provides better fit for the above data set as it provides minimum AIC, BIC, CAIC, HQIC.

Data Set 3

The 3rd data set is reported by Bekker et al. (2000), which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 46 patients are: 0.047,0.115, 0.121,0.132,0.164,0.197,0.203,0.260,0.282,0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.033. We have compared KL2PFD with the Kumarswamy Marshall-Olkin family of distribution (Kw-MO) (see Morad et al. 2015).

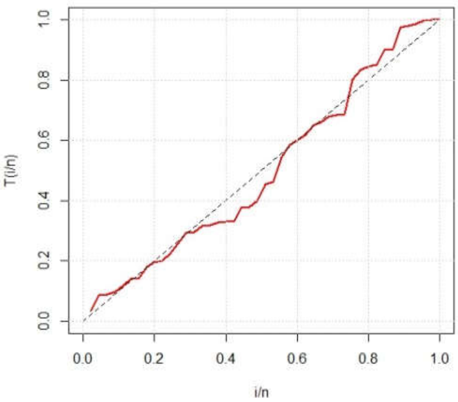

Figure 8 TTT plot for chemotherapy treatment

The TTT plot is displayed in Figure 8 which indicates that the HRF associated with the data set has a bathtub shape, so we can easily fit KL2PFD on the Chemotherapy data. The proposed model KL2PFD is showing better results as compared to the (Morad et al. 2015) and the other competitive models by providing smallest AIC, BIC, CAIC and HQIC for the given data.

Table 7 Statistics of chemotherapy treatment data

Models	AIC	BIC	CAIC	HQIC
KL2PFD	**105.5555**	**106.6081**	**112.6003**	**108.1534**
KwMO-W	119.134	120.672	128.167	122.501
BW	123.995	124.995	131.222	126.689
KwW	124.189	125.189	131.416	126.884
EW	122.087	122.673	127.507	124.108
MOW	121.716	122.301	127.136	123.736

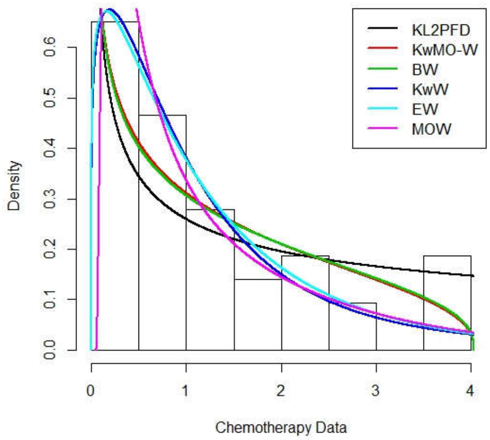

Figure 9 Estimated density plot for chemotherapy data

Data Set 4

The data have 63 observations about the strengths of 1.5 cm glass fiber. It was obtained from workers at the UK National Physical Laboratory. The same data have also been examined by (Smith and Naylor 1987). The data are: 0.550, 0.740, 0.770, 0.811, 0.841, 0.932, 1.040, 1.111, 1.130, 1.240, 1.251, 1.270, 1.281, 1.290, 1.301, 1.360, 1.390, 1.420, 1.480, 1.480, 1.490, 1.490, 1.500, 1.500, 1.510, 1.520, 1.530, 1.540, 1.550, 1.550, 1.580, 1.590, 1.600, 1.610, 1.610, 1.610, 1.610, 1.621, 1.621, 1.630, 1.640, 1.660, 1.660, 1.660, 1.670, 1.680, 1.680, 1.690, 1.701, 1.701, 1.730, 1.760, 1.760, 1.770, 1.780, 1.810, 1.820, 1.840, 1.840, 1.890, 2.000, 2.010, 2.240.

We have compared KL2PFD with the modified Burr III Weibull distribution by Arifa et al. (2017), modified Burr III and Weibull models.

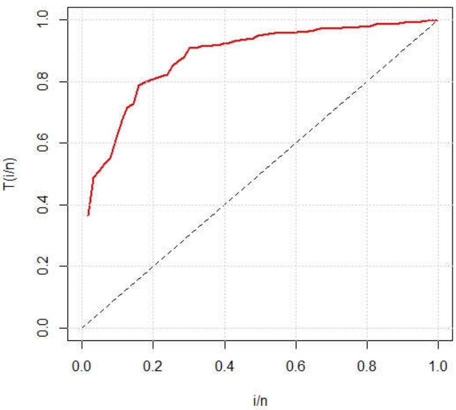

Figure 10 TTT plot for glass fiber data

The TTT plot is displayed in Figure 10, which indicates that the HRF associated with the data set has an increasing shape, since the plot shows a first concave curvature. So, we can easily fit KL2PFD on the carbon fibers data. The proposed model KL2PFD is showing better results as compare to the other competitive models by providing smallest AIC, BIC, CAIC and HQIC for the given data.

Table 8 Statistics of glass fiber data

Models	AIC	BIC	CAIC	HQIC
KL2PFD	**23.57246**	**29.95387**	**23.98626**	**26.07796**
MBIII Weibull	32.6	43.3	33.7	28.432
MBIII	30.0	36.4	30.4	31.249
Weibull	34.4	38.7	34.6	33.1565

Table 8 provides minimum values of AIC, BIC, CAIC and HQIC for KL2PFD which means that proposed model provides better fit for the data.

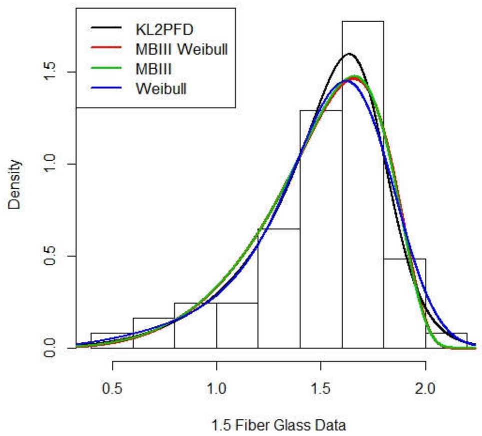

Figure 11 Estimated density plot for glass fiber data

9. Conclusions

We have proposed a new distribution called Kumaraswamy Lehmann-2 power function distribution (KL2PFD). This distribution can have applications in the fields of reliability, economics, actuaries and survival analysis. We have studied the properties of the new distribution including moments, survival function, hazard function, inverse moments, Shanon entropy, conditional moments, Lorenz curve, incomplete moments and order Statistics. We have also characterized the distribution by conditional moments (right and left truncated mean) and doubly truncated mean (DTM). Four different data sets from different scenarios of applied sciences are used to show the efficiency of the proposed model over the already available models. It is hoped that the findings of this paper will be useful for researchers in different field of applied sciences.

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