Chinese Journal of Physics 77 (2022) 1827–1842

Contents lists available at ScienceDirect

Chinese Journal of Physics

journal homepage: www.elsevier.com/locate/cjph

Class I stars in the background of $f(T,\tau)$ cosmological model

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ARTICLE INFO

Keywords:
$f(T,\tau)$ modified gravity
Compact stars
Karmarkar condition

ABSTRACT

The present paper is designed to study the structure of some compact objects such as $PSRJ1614-2230$, $LMXB\ 4U1608-52$, $CenX-3$, $EXO1785-248$ and $SMCX-1$ in the framework of $f(T,\tau)$ gravity, where τ and T represent the trace of energy–momentum tensor and torsion scalar, respectively. For this work, we use anisotropic fluid distribution filled in spherical symmetric geometry providing the interior of star model and utilize the off-diagonal tetrad components for deriving the corresponding set of field equations. We construct stellar structure by using a well-known model of $f(T,\tau)$ gravity, given by $f(T,\tau)=\alpha T(r)^2+\beta\tau(r)+\phi$, where α, β and ϕ are arbitrary constants. This model has interesting cosmological properties, in addition to providing an effective dark energy sector explanation (Harko et al., 2014). For checking viability of the constructed model, we set $\alpha<0$, $\beta<0$ and discuss physical and stability features like anisotropy, energy conditions, EoS parameters, TOV equation, redshift, compactness and mass function of strange stars. It is found that these features comprehensively match with the realistic nature of compact stars in the realm of $f(T,\tau)$ gravity.

1. Introduction

Accelerated expansion of our cosmos and the search of its responsible factor has drawn considerable attention of the cosmologists in modern astrophysics [1,2]. In this respect, different observational data sets are available in literature which directly or indirectly support this phenomena of cosmic acceleration [3–5]. It has been argued that a dominant unusual kind of energy, covering almost 72% of the total cosmos matter distribution, is a major cause of this cosmos expansion. This obscure natured force having strong negative pressure is named as dark energy (DE). Although general theory of relativity (GR) is regarded as a primary gravitational framework but it malfunction to produce adequate results while analyzing the phenomenon of cosmic acceleration. The exploration of a satisfactory candidate of DE has posed a substantial challenge for researchers and as a result, numerous approaches have been presented in literature. These approaches either extend the matter part or the curvature sector of the action of GR and hence labeled as extended matter proposals or modified gravity theories, respectively. A careful comparison of both techniques suggested that the curvature part extension, i.e., the modified theories are more useful in investigating different cosmic aspects. For the review of these interesting candidates, one can refer to the literature [6].

The $f(T)$ gravity is a well-known gravitational framework which is based on "teleparallel" approach (initially provided a theory equivalent to GR and is known as TEGR) [7]. The TEGR and its $f(T)$ extension both involve the Weitzenbock connection as a fundamental tool where the gravitational part is described by the torsion only and consequently is free from curvature. Using a good tetrad (non-diagonal), adequately correlates with the isotropic spherically symmetric metric. In $f(T)$ gravity, dynamical equations

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https://doi.org/10.1016/j.cjph.2021.12.029
Received 4 October 2021; Received in revised form 13 December 2021; Accepted 23 December 2021
Available online 4 February 2022
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display an extra degree of freedom as its matter distribution is constructed under non-variant Lorentz transformation [8]. Thus $f(T)$ gravity is an approximate generalization of TEGR promising the gravitational part of Lagrangian as a function of torsion T [8,9]. On the cosmological landscape, $f(T)$ theory of gravity has been used by several researchers and proved to be very promising, for reference, one can see the literature [10]. In parallel to the ideas used for the extension of Einstein–Hilbert action, one can also utilize same ideas to generalize TEGR as well as $f(T)$ theories. Following a similar pattern, some interesting general teleparallel frameworks have been proposed [11] like $f(T,T_G)$ theory, $f(T,B)$ gravity, $f(T,L_m)$ framework and higher-order derivative of torsion based teleparallel theory etc. In addition to this, another idea was to combine both modified matter and geometry approaches together and develop a single generalized formulation. Based on this idea, some interesting extensions of GR includes $f(R,T)$ gravity [12], $f(R,L_m)$ [13] and $f(R,T,Q)$ theories [14], where T is the trace of energy–momentum tensor. Likewise, an extension of $f(T)$ gravity based on such idea was formulated namely $f(T,\tau)$ theory (τ is the energy–momentum tensor trace) and found to be very significant on cosmological grounds [15–17].

Study of compact stars is a hot topic of discussion for modern researchers in the framework of different modified theories. Compact stars are hugely dense than ordinary stars having small radius and bulky mass. Compact star come into existence as a result of evolutionary stage of an ordinary star when the fusion processes start inside its core and the star do not balance the gravitational forces present in it and hence the star collapse under its own weight. Highly dense objects in astrophysics containing strange quark matter of (u), (d) and (s) [18–20] are known as strange stars. On the basis of large amount of observational data available, it is shown that $HerX-1$, $4U1820-30$ and $PSRJ1614-2230$ are good contestants of strange star [21–23].

At first, Schwarzschild [24,25] presented the interior solution of a stellar object filled with isotropic fluid (in which radial and tangential pressure are equal). Later, Ruderman [26] studied the anisotropic $(p_r \neq p_t)$ fluid distribution of the stellar structures. In the framework of GR, anisotropic effects on spherically symmetric compact stellar distribution have been described by numerous authors including Ivanov [27], Mak and Harko [28], Hossein et al. [29], Kalam et al. [30], Bahar [31] and Maurya et al. [32]. In the context modified gravity theories, several investigators have conducted significant discussions about the existence and construction of compact stars by exploring their theoretical and analytical features. In order to see some examples of compact star models proposed in modified gravity theories, one can see the literature [33–39].

In the construction of compact stellar objects, one of most interesting and successful tools is the application of Karmarkar condition. The Karmarkar condition suggests a relationship between the metric components and then by considering one of the gravitational components of metric say g_{tt}, the other component g_{rr} can be calculated easily. In this way, a 4-dimensional Riemanian space–time can be embedded into a 5-dimensional pseudo–Euclidean space–time (known as embedding class I metric) without altering its inner properties. It has been argued that embedding class I is a necessary condition to get solutions via Karmarkar condition and the solutions obtained, in this way, are termed as embedding class I solutions. In the papers [40,41], authors have elaborated the process of embedding a 4-dimensional Riemanian metric into 5-dimensional Euclidean metric. In the recent past, plenty of work so far have been done by different researchers on the subject of anisotropic compact stars existence and construction in different gravitational frameworks like Pandya et al. [42], Abbas et al. [43], Jaya et al. [44], Pandya and Thomas [45], Mustafa et al. [46], Abellán et al. [47] and Ramos et al. [48] by using Karmarkar condition. Likewise, Maurya and his collaborators [49] presented some interesting models on this subject and found significant results. In the context of $f(T,\tau)$ gravity, Salako et al. [50] discussed the existence of compact star models using diagonal tetrad obtaining some interesting results. In another study, Saleem et al. [51] constructed model representing LMC X-4 and Vela X-1 compact stars using specific linear function in $f(T,\tau)$ gravity and described the physical significance of their models.

Motivated by these good antecedents, here we shall study anisotropic models representing some compact objects, such as $PSRJ1614-2230$, $LMXB4U1608-52$, $CenX-3$, $EXO1785-248$ and $SMCX-1$ via embedding approach, in $f(T,\tau)$ gravity scenario by using static spherically symmetric metric as interior geometry. So, the article is organized as follows: In the upcoming Section 2, we shall explain the preliminary concepts of $f(T,\tau)$ formulation and also introduce the well-known Karmarkar condition to obtain the generalized solution of the respective field equations. In Section 3, we shall describe the matching conditions of inner and outer geometries of compact star and calculate the values of unknown constants. In Section 4, we shall explore the viability of the constructed structure by analyzing some interesting and essential features of compact stars, like the impact of local anisotropies, validity of energy conditions, EoS parameters, causality and stability conditions. Next, in Section 5 the redshift, adiabatic index, compactness and mass function of the model are analyzed by means of a graphical study. Finally, Section 6 concludes the work.

2. Basics of $f(T,\tau)$ gravity, its field equations and an introduction to embedding approach

In this section, we shall present a brief introduction to $f(T,\tau)$ modification of teleparallel theory and its resulting field equations for spherically symmetric space time. Here we shall also list some essential assumptions taken for this work. Similar to $f(R,T)$ gravity which is considered as one of the most interesting and successful extension of $f(R)$ and hence of GR, the teleparallel theory can be extended to a viable form by including the trace of energy–momentum tensor term in its $f(T)$ generalization. The TEGR and its modifications are based on a key ingredient of curvature-less Wietzenbock connection and the orthogonal tetrad field components in tangential space manifold. The action of $f(T,\tau)$ gravity is defined by the equation [15,50]

$$s = \int dx^4 h \left[\frac{1}{2k^2} f(T,\tau) + \mathcal{L}_{(M)}\right], \tag{1}$$

where $\mathcal{L}_{(M)}$ is the ordinary matter Lagrangian density and f is an arbitrary function depending on upon the trace of energy–momentum tensor denoted by τ and the torsion scalar T. Basically, L_M represents the ordinary matter Lagrangian density and the corresponding energy–momentum tensor $\tau_{\xi\psi}$ is defined as follows [50]

$$\tau_{\xi\psi} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_M)}{\delta g^{\xi\psi}}$$

and then the trace of energy–momentum tensor is defined as $\tau = g^{\xi\psi}\tau_{\xi\psi}$. Since L_M depends on the metric components only, not its derivatives, therefore we can write

$$\tau_{\xi\psi} = g_{\xi\psi}L_M - 2\frac{\partial L_M}{\partial g^{\xi\psi}}.$$

In the present work, we shall consider the anisotropic fluid as ordinary matter contents. Also, here $h = det\left(h^A_\mu\right) = \sqrt{-g}$ and $k^2 = 8\pi G = 1$. The variation of the action given above with respect to the tetrad results into the following set of general field equations:

$$h_i{}^\gamma S_\gamma{}^{\mu\nu} f_{TT}\partial_\mu T + h_i{}^\gamma S_\gamma{}^{\mu\nu} f_{T\tau}\partial_\mu \tau + h^{-1}\partial_\mu(hh_i{}^\gamma S_\gamma{}^{\mu\nu})f_T - h_i{}^\lambda T^\gamma{}_{\mu\lambda}S_\gamma{}^{\nu\mu}f_T - \frac{1}{4}h_i{}^\nu f + f_T\omega^\gamma{}_{i\mu}S_\gamma{}^{\mu\nu}$$
$$-\frac{f_\tau}{2}(h_i{}^\gamma \tau_\gamma{}^\nu + p_t h_i{}^\nu) = -4\pi h_i{}^\gamma \tau_\gamma{}^\nu, \tag{2}$$

where τ^ν_γ is the notation for energy–momentum tensor of ordinary matter. These set of dynamical equations involve the first and second-order derivatives of the function f which are represented by the notations $f_T = \frac{\partial f}{\partial T}$, $f_{TT} = \frac{\partial^2 f}{\partial T^2}$, $f_\tau = \frac{\partial f}{\partial \tau}$ and $f_{T\tau} = \frac{\partial^2 f}{\partial T \partial \tau}$. It would be worthy to mention here that we have taken the spin connection as zero in the beginning, i.e., $\omega^\gamma{}_{i\mu} = 0$. The building blocks of TEGR namely torsion, contorsion and super potential tensors, which are also present in Eq. (2), can be written, respectively, as

$$T^\lambda_{\mu\eta} = h_\vartheta{}^\lambda(\partial_\mu h^\vartheta{}_\eta - \partial_\eta h^\vartheta{}_\mu), \tag{3}$$
$$K^{\mu\eta}_\lambda = -\frac{1}{2}\left(T^{\mu\eta}{}_\lambda - T^{\eta\mu}{}_\rho - T_\lambda{}^{\mu\eta}\right), \tag{4}$$
$$S_\lambda{}^{\mu\eta} = \frac{1}{2}\left(K^{\mu\eta}{}_\lambda + \delta^\mu{}_\lambda T^{\gamma\mu}{}_\gamma - \delta^\eta{}_\lambda T^{\gamma\mu}{}_\gamma\right). \tag{5}$$

The density of teleparallel lagrangian is defined by torsion scalar and is given by

$$T = T^\lambda{}_{\kappa\eta}S_\lambda{}^{\kappa\eta}. \tag{6}$$

For the construction of compact star structure in this gravity, let us consider the spacetime geometry which exhibits the symmetry that is closest to the one available in nature, the spherically symmetry, and is defined by the following line element:

$$ds^2 = e^{\zeta(r)}dt^2 - e^{\chi(r)}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{7}$$

where $\zeta(r)$ and $\chi(r)$ are the radial coordinate dependent unknown functions. It is interesting to mention here that the choice of tetrad components play a vital role in setting up the dynamical equations and consequently, in the construction of stellar structures. So, the selection of good tetrad components is mandatory in this respect. For the present work, we have considered the off-diagonal (good) tetrad components and the corresponding $f(T,\tau)$ gravitational field equations can be written as [52–54]

$$h^\eta_\gamma = \begin{pmatrix} e^{\frac{\zeta(r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\chi(r)}{2}}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & e^{\frac{\chi(r)}{2}}\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & e^{\frac{\chi(r)}{2}}\cos\theta & -r\sin\theta & 0 \end{pmatrix}. \tag{8}$$

For this choice of tetrad component, the value of its determinant denoted by $h = h^\eta_\gamma$ takes the form as $e^{\zeta(r)+\chi(r)}r^2\sin\theta$. Let us consider the interior of compact star is filled with an ordinary matter exhibiting anisotropic properties and is defined by the energy–momentum tensor

$$\tau^{(m)}_{\xi\psi} = (\rho + p_t)u_\xi u_\psi - p_t g_{\xi\psi} + (p_r - p_t)v_\xi v_\psi, \tag{9}$$

where the velocities satisfy the relations $u_\xi = e^{\frac{\mu}{2}}\delta^0_\xi$ and $v_\xi = e^{\frac{\zeta}{2}}\delta^1_\xi$, and the symbols ρ, p_r and p_t, refer to the energy density, radial and tangential pressures, respectively. This further can be re-written as in the diagonal form: $\tau^{(m)}_{\xi\psi} = [\rho, -p_r, -p_t, -p_t]$ and consequently, the trace of energy–momentum tensor will take the form as follows

$$\tau = \delta^\nu_\mu T^\mu_\nu = \rho - p_r - 2p_t. \tag{10}$$

Using the tetrad components, the torsion scalar and its radial rate of change denoted by $T'(r)$ can be written as

$$T(r) = \frac{2e^{-\chi(r)}\left(e^{\frac{\chi(r)}{2}} - 1\right)\left(e^{\frac{\chi(r)}{2}} - r\zeta'(r) - 1\right)}{r^2}, \tag{11}$$

$$T' = -\frac{4e^{-\chi(r)}\left(e^{\frac{\chi(r)}{2}}-1\right)\left(e^{\frac{\chi(r)}{2}}-r\zeta'(r)-1\right)}{r^3}+\frac{e^{-\frac{\chi(r)}{2}}\chi'(r)\left(e^{\frac{\chi(r)}{2}}-r\zeta'(r)-1\right)}{r^2}$$
$$-\frac{2e^{-\chi(r)}\left(e^{\frac{\chi(r)}{2}}-1\right)\chi'(r)\left(e^{\frac{\chi(r)}{2}}-r\zeta'(r)-1\right)}{r^2}+\frac{2e^{-\chi(r)}\left(e^{\frac{\chi(r)}{2}}-1\right)\left(\frac{1}{2}e^{\frac{\chi(r)}{2}}\chi'(r)-r\zeta''(r)-\zeta'(r)\right)}{r^2}. \quad (12)$$

The dynamical field equations of $f(T,\tau)$ gravity (2) for the metric (7) are given by

$$\rho = -\frac{e^{-\frac{\chi(r)}{2}}\left(e^{-\frac{\chi(r)}{2}}-1\right)\left(f_{TT}T'+f_{T\tau}\tau'\right)}{r}-\frac{1}{2}f_T\left(-\frac{e^{-\chi(r)}\left(1-r\chi'(r)\right)}{r^2}-\frac{1}{r^2}+\frac{T(r)}{2}\right)+\frac{f}{4}+\frac{1}{2}f_\tau(\mathrm{pt}+\rho), \quad (13)$$

$$p_r = \left(\frac{e^{-\chi(r)}\left(r\zeta'(r)+1\right)}{r^2}-\frac{1}{r^2}+\frac{T(r)}{2}\right)\frac{f_T}{2}-\frac{f}{4}-\frac{1}{2}f_\tau(\mathrm{pt}-\mathrm{pr}), \quad (14)$$

$$p_t = \frac{1}{2}e^{-\chi(r)}\left(-\frac{e^{\frac{\chi(r)}{2}}}{r}+\frac{\zeta'(r)}{2}+\frac{1}{r}\right)\left(f_{TT}T'+f_{T\tau}\tau'\right)+\left[e^{-\chi(r)}\left(\left(\frac{\zeta'(r)}{4}+\frac{1}{2r}\right)\left(\zeta'(r)-\chi'(r)\right)+\frac{\zeta''(r)}{2}\right)\right.$$
$$\left.+\frac{T(r)}{2}\right]\frac{f_T}{2}-\frac{f}{4}, \quad (15)$$

where we have used off-diagonal tetrad components (8) and energy–momentum tensor (9). It can be easily checked that the above system of equations is not closed (three equations are involving 6 unknowns to be computed). In order to get solutions, we need to assume three conditions for the involved unknowns. In the present work, we shall utilize a well-famed model of $f(T,\tau)$ gravity given by $f(T,\tau)=\alpha T^m(r)+\beta\tau(r)+\phi$, in which the constants α, β and ϕ are all real and arbitrary while the constant m is any real number such that $m\neq 0$. It is interesting to mention here that for the choice: $\alpha=1$, $m=1$, $\beta=\phi=0$, one can retrieve the standard TEGR framework. In the present work, we shall fix $m=2$ in all subsequent calculations (i.e., $f(T,\tau)=\alpha T^2(r)+\beta\tau(r)+\phi$). The main motivation behind this choice, are the good cosmological properties and the capacity to explain the effective dark energy sector by interpreting it as a quintessence or phantom-like field [15]. All these features studied on a FRW background. Given the versatility of the selected model, the question naturally arises whether it is possible to describe a well-behaved stellar interior using such $f(T,\tau)$ functional. Then, inserting all previously defined relations into the field equations (13)–(15), the final expressions of density and pressure functions can be can written as

$$\rho = \frac{e^{-2\chi(r)}}{4\left(\beta^2-3\beta+2\right)r^4}\Big[72\alpha-192\alpha e^{\frac{\chi(r)}{2}}+176\alpha e^{\chi(r)}-64\alpha e^{\frac{3\chi(r)}{2}}+8\alpha e^{2\chi(r)}-60\alpha\beta+160\alpha\beta e^{\frac{\chi(r)}{2}}-144\alpha\beta e^{\chi(r)}$$
$$+48\alpha\beta e^{\frac{3\chi(r)}{2}}-4\alpha\beta e^{2\chi(r)}+2r^4\phi e^{2\chi(r)}-\beta r^4\phi e^{2\chi(r)}-4\alpha r\left(e^{\frac{\chi(r)}{2}}-1\right)\left(3\beta\left(e^{\frac{\chi(r)}{2}}-3\right)+8\right)\zeta'(r)$$
$$+4\alpha r^2\left(e^{\frac{\chi(r)}{2}}-1\right)\left((\beta-2)e^{\frac{\chi(r)}{2}}+2\right)\zeta'(r)^2-4\alpha(3\beta-4)r\left(e^{\frac{\chi(r)}{2}}-1\right)^2\chi'(r)\left(r\zeta'(r)+1\right)-32\alpha r^2\zeta''(r)$$
$$+64\alpha r^2e^{\frac{\chi(r)}{2}}\zeta''(r)-32\alpha r^2e^{\chi(r)}\zeta''(r)+24\alpha\beta r^2\zeta''(r)-48\alpha\beta r^2e^{\frac{\chi(r)}{2}}\zeta''(r)+24\alpha\beta r^2e^{\chi(r)}\zeta''(r)\Big], \quad (16)$$

$$p_r = \frac{e^{-2\chi(r)}}{4\left(\beta^2-3\beta+2\right)r^4}\Big[24\alpha-64\alpha e^{\frac{\chi(r)}{2}}+48\alpha e^{\chi(r)}-8\alpha e^{2\chi(r)}-36\alpha\beta+96\alpha\beta e^{\frac{\chi(r)}{2}}-80\alpha\beta e^{\chi(r)}+16\alpha\beta e^{\frac{3\chi(r)}{2}}$$
$$+4\alpha\beta e^{2\chi(r)}-2r^4\phi e^{2\chi(r)}+\beta r^4\phi e^{2\chi(r)}-4\alpha r\left(e^{\frac{\chi(r)}{2}}-1\right)\left(-11\beta+3(3\beta-4)e^{\frac{\chi(r)}{2}}+12\right)\zeta'(r)-4\alpha r^2\left(e^{\frac{\chi(r)}{2}}-1\right)$$
$$\times\left(-4\beta+(\beta-2)e^{\frac{\chi(r)}{2}}+6\right)\zeta'(r)^2-4\alpha\beta r\left(e^{\frac{\chi(r)}{2}}-1\right)^2\chi'(r)\left(r\zeta'(r)+1\right)+8\alpha\beta r^2\zeta''(r)-16\alpha\beta r^2e^{\frac{\chi(r)}{2}}\zeta''(r)$$
$$+8\alpha\beta r^2e^{\chi(r)}\zeta''(r)\Big] \quad (17)$$

$$p_t = \frac{e^{-2\chi(r)}}{4(\beta-2)(\beta-1)r^4}\Big[-24\alpha+64\alpha e^{\frac{\chi(r)}{2}}-48\alpha e^{\chi(r)}+8\alpha e^{2\chi(r)}+12\alpha\beta-32\alpha\beta e^{\frac{\chi(r)}{2}}+16\alpha\beta e^{\chi(r)}+16\alpha\beta e^{\frac{3\chi(r)}{2}}$$
$$-12\alpha\beta e^{2\chi(r)}-2r^4\phi e^{2\chi(r)}+\beta r^4\phi e^{2\chi(r)}-4\alpha r^2\left(e^{\frac{\chi(r)}{2}}-1\right)\left(-\beta+(2\beta-3)e^{\frac{\chi(r)}{2}}+3\right)\zeta'(r)^2+4\alpha(\beta-1)r^3$$
$$\times\left(e^{\frac{\chi(r)}{2}}-1\right)\zeta'(r)^3-4\alpha r\chi'(r)\Big[(\beta-1)r^2\left(2e^{\frac{\chi(r)}{2}}-3\right)\zeta'(r)^2-r\left(e^{\frac{\chi(r)}{2}}-1\right)\left(-8\beta+(2\beta-3)e^{\frac{\chi(r)}{2}}+9\right)\zeta'(r)$$
$$-(5\beta-6)\left(e^{\frac{\chi(r)}{2}}-1\right)^2\Big]+24\alpha r^2\zeta''(r)-48\alpha r^2e^{\frac{\chi(r)}{2}}\zeta''(r)+24\alpha r^2e^{\chi(r)}\zeta''(r)-16\alpha\beta r^2\zeta''(r)+32\alpha\beta r^2e^{\frac{\chi(r)}{2}}\zeta''(r)$$
$$-16\alpha\beta r^2e^{\chi(r)}\zeta''(r)+4\alpha r\left(e^{\frac{\chi(r)}{2}}-1\right)\zeta'(r)\left(2+4(\beta-1)e^{\chi(r)}-3\beta+(\beta+2)e^{\frac{\chi(r)}{2}}+4(\beta-1)r^2\zeta''(r)\right)\Big]. \quad (18)$$

The well-known Karmarkar condition [40] facilitates the process of linking two gravitational potential components of metric function, yielding the most convenient technique to calculate the metric components if any one of these components is known. For embedding class-I solutions, the condition considered by Karmarkar comes from the following necessary and sufficient conditions that any spherically symmetric space–time (static and non-static) must satisfy in order to be of class-I. Those are:

- A system of symmetric quantities $b_{\mu\nu}$ must be established, such that

$$R_{\mu\nu\alpha\beta} = \epsilon\,\left(b_{\mu\alpha}\,b_{\nu\beta} - b_{\mu\beta}\,b_{\nu\alpha}\right) \quad \text{(Gauss's equation)}, \tag{19}$$

  where $\epsilon = \pm 1$ whenever the normal to the manifold is space-like (+1) or time-like (−1).
- The system $b_{\mu\nu}$ must satisfy the differential equations

$$\nabla_\alpha b_{\mu\nu} - \nabla_\nu\, b_{\mu\alpha} = 0 \quad \text{(Codazzi's equation)}. \tag{20}$$

  This form of Eq. (20) is implied by Eq. (19) [55].

The required components of Riemanian Tensor for the interior spacetime (7) are given by

$$\begin{aligned} R_{1414} &= -\frac{1}{4}e^{\zeta(r)}\left(-\zeta'(r)\chi'(r)+\zeta'^2(r)+2\zeta''(r)\right), \quad R_{2323} = -e^{-\chi(r)}r^2\sin^2\theta(e^{\chi(r)}-1), \quad R_{1212} = \frac{1}{2}r\chi', \\ R_{3434} &= -\frac{1}{2}r\sin^2\theta\zeta'(r)e^{\zeta(r)-\chi(r)}. \end{aligned} \tag{21}$$

So, by using the set of Eqs. (21) into Eq. (19) one gets

$$\begin{aligned} &b_{14}\,b_{22} = R_{1343} = 0; \quad b_{14}\,b_{33} = R_{1242} = 0; \\ &b_{44}\,b_{33} = R_{4343}; \quad b_{44}\,b_{22} = R_{4242}; \quad b_{11}\,b_{33} = R_{1313}; \\ &b_{22}\,b_{33} = R_{2323}; \quad b_{11}\,b_{22} = R_{1212}; \quad b_{44}\,b_{11} = R_{4141}. \end{aligned} \tag{22}$$

The above relations leads to

$$\left(b_{44}\right)^2 = \frac{\left(R_{4242}\right)^2}{R_{2323}}\,sin^2\theta, \quad \left(b_{11}\right)^2 = \frac{\left(R_{1212}\right)^2}{R_{2323}}\,sin^2\theta, \quad \left(b_{22}\right)^2 = \frac{R_{2323}}{sin^2\theta}, \quad \left(b_{33}\right)^2 = sin^2\theta\,R_{2323}. \tag{23}$$

Upon replacing (23) into expression (22), one gets:

$$R_{1212}\,R_{1313} = R_{4141}\,R_{2323}, \tag{24}$$

subject to $R_{2323} \neq 0$. It should be noted that Eq. (22) satisfies Codazzi's equation (20). On the other hand, in the case of a general non-static spherically symmetric space–time, the second and last equality in (22) become:

$$b_{41}\,b_{22} = R_{1242} \quad \text{and} \quad b_{44}\,b_{11} - \left(b_{41}\right)^2 = R_{4141}, \tag{25}$$

where $\left(b_{41}\right)^2 = sin^2\theta\left(R_{1242}\right)^2/R_{2323}$. So, the class-I condition is given by [40]

$$R_{4242}\,R_{1313} = R_{4141}\,R_{2323} + R_{1242}\,R_{1343}, \tag{26}$$

or equivalently

$$R_{1414} = \frac{R_{1313}R_{2424} - R_{1242}R_{1343}}{R_{2323}}; \quad R_{2323} \neq 0. \tag{27}$$

By the replacement of these non-vanishing Riemanian tensor components into Eq. (27), we get the following differential equation relating the two metric components:

$$\zeta'(r) + \frac{2\zeta''(r)}{\zeta'(r)} = \frac{e^{\chi(r)}\chi'(r)}{e^{\chi(r)}-1}. \tag{28}$$

Integration of the above differential equation (28) results into an expression of $e^{\zeta(r)}$ (g_{tt} metric component) and is given by

$$e^{\zeta(r)} = \left(A + B\int\sqrt{e^{\chi(r)}-1}\,dr\right)^2. \tag{29}$$

Here A and B are constants of integration.. Based on this idea, many interesting works are available in literature [56] where class-I solutions have been proposed in various contexts including gravitational decoupling approach.

Let us consider an interesting model of metric component g_{rr} (already available in literature [57]) and is given by

$$e^{\chi(r)} = 1 + ar^2e^{n\sin^{-1}\left(br^2+c\right)}, \tag{30}$$

where $a,\ b,\ c$ are all arbitrary non-negative constants and $n \in \mathbb{R};\ \ n \geq 0$. Consequently, the g_{tt} metric potential will take the form as

$$e^{\zeta(r)} = \left(\frac{B\left(n\sqrt{1-\left(br^2+c\right)^2}+2br^2+2c\right)\sqrt{ar^2e^{n\sin^{-1}\left(br^2+c\right)}}}{b\left(n^2+4\right)r} + A\right)^2. \tag{31}$$

Under all these assumptions along with the metric potentials, the expressions of ρ, p_r and p_t will turn out to be

$$\begin{aligned}
\rho =\ & \frac{1}{4\left(\beta^2-3\beta+2\right) r^4 f_1^2(r)} \Big[72\alpha - 192\alpha f_1^{\frac{1}{2}}(r) + 176\alpha f_1(r) - 64\alpha f_1^{\frac{3}{4}}(r) + 8\alpha f_1^2(r) - 64\alpha b B\left(n^2+4\right) r^2 f_3(r) g_1(r) \\
&+ \frac{128\alpha b B\left(n^2+4\right) r^2 f_3^2(r)}{\sqrt{\frac{f_3^2(r)}{f_3^2(r)+1}}} g_1(r) - 64\alpha b B\left(n^2+4\right) r^2 f_3(r) f_1(r) g_1(r) - 60\alpha\beta + 160\alpha\beta f_1^{1/2}(r) - 144\alpha\beta f_1(r) \\
&+ 48\alpha\beta f_1^{3/2}(r) - 4\alpha\beta f_1^2(r) + 48\alpha b\beta B\left(n^2+4\right) r^2 f_3(r) g_1(r) - \frac{96\alpha b\beta B\left(n^2+4\right) r^2 f_3^2(r)}{\sqrt{\frac{f_3^2(r)}{f_3(r)+1}}} g_1(r) + 48\alpha b\beta B\left(n^2+4\right) \\
&\times r^2 f_3(r) g_1(r) + \Big[8\alpha(3\beta-4) r\left(f_2(r)+bnr^2\right) f_3(r)\left(f_1^{1/2}(r)-1\right)^2 \Big[Ab\left(n^2+4\right) f_2(r) f_3(r) + B f_3^2(r) \\
&\times \Big[-n\left(b^2r^4+2bcr^2+c^2-1\right)+2bn^2r^2 f_2(r) + 2\left(5br^2+c\right) f_2(r)\Big]\Big]\Big] \Big/ \Big[\left(b^2r^4+2bcr^2+c^2-1\right) f_1(r) \\
&\times \Big[B\left(n f_2(r)+2br^2+2c\right) f_3(r) + Ab\left(n^2+4\right) r\Big]\Big] + \left(\beta f_1^{1/2}(r) - 2 f_1^{1/2}(r) + 2\right) g_2(r) - \left(3\beta\left(f_1^{1/2}(r)-3\right)+8\right) \\
&\times g_3(r) + 2r^4\phi f_1^2(r) - \beta r^4 \phi f_1^2(r)\Big],
\end{aligned} \tag{32}$$

$$\begin{aligned}
p_r =\ & \frac{1}{4r^2 f_1^2(r)(\beta^2-3\beta+2)} \Big[24\alpha - 64\alpha f_1^{\frac{1}{2}}(r) + 48\alpha f_1(r) - 8\alpha f_1^2(r) - 36\alpha\beta + 96\alpha\beta f_1^{\frac{1}{2}}(r) - 80\alpha\beta f_1(r) + 16\alpha\beta f_1^{\frac{3}{2}}(r) \\
&+ a\alpha\beta f_1^2(r) + 16\alpha b\beta B\left(n^2+4\right) r^2 f_3(r) g_1(r) - \frac{32\alpha b\beta B\left(n^2+4\right) r^2 f_3^2(r)}{\sqrt{\frac{f_3^2(r)}{f_1(r)}}} g_1(r) + 16\alpha b\beta B\left(n^2+4\right) r^2 f_3(r) f_1(r) g_1(r) \\
&+ \Big[8r\alpha\beta f_3(r)(f_1^{\frac{1}{2}}(r)-1)^2\left(f_2(r)+bnr^2\right) \Big[Ab\left(n^2+4\right) f_2(r) f_3(r) + B\Big[-n\left(b^2r^4+2bcr^2+c^2-1\right)+2bn^2r^2 f_2(r) \\
&+ 2\left(5br^2+c\right) f_2(r)\Big] f_3^2(r)\Big]\Big] \Big/ \Big[\left(b^2r^4+2bcr^2+c^2-1\right) f_1(r)\left(B\left(n f_2(r)+2br^2+2c\right) f_3(r) + Ab\left(n^2+4\right) r\right)\Big] \\
&- \left(9\beta f_1^{\frac{1}{2}}(r) - 12 f_1^{\frac{1}{2}}(r) - 11\beta + 12\right) g_3(r) - \left(\beta\left(f_1^{\frac{1}{2}}(r)-4\right) - 2 f_1^{\frac{1}{2}}(r) + 6\right) g_2(r) - \beta r^4\phi f_1^2(r) + 2r^4\phi f_1^2(r)\Big],
\end{aligned} \tag{33}$$

$$\begin{aligned}
p_t =\ & \frac{1}{4r^4 f_1^2(r)(\beta-2)(\beta-1)} \Big[-24\alpha + 64\alpha f_1^{1/2}(r) - 48\alpha f_1(r) + 8\alpha f_1^2(r) + 48\alpha b B\left(n^2+4\right) r^2 f_3(r) g_1(r) \\
&- \frac{96\alpha b B\left(n^2+4\right) r^2 f_3^2(r)}{\sqrt{\frac{f_3^2(r)}{f_3^2(r)+1}}} g_1(r) + 48\alpha b B\left(n^2+4\right) r^2 f_3(r) f_1(r) g_1(r) \\
&+ \frac{32\alpha b^3(\beta-1) B^3\left(n^2+4\right)^3 r^6 f_3^3(r)\left(f_1^{1/2}(r)-1\right)}{\left(B\left(n f_2(r)+2br^2+2c\right) f_3(r) + Ab\left(n^2+4\right) r\right)^3} + 12\alpha\beta - 32\alpha\beta f_1^{1/2}(r) + 16\alpha\beta f_1(r) + 16\alpha\beta f_1^{3/2}(r) - 12\alpha\beta \\
&\times f_1^2(r) - 32\alpha b\beta B\left(n^2+4\right) r^2 f_3(r) g_1(r) + \frac{64\alpha b\beta B\left(n^2+4\right) r^2 f_3^2(r)}{\sqrt{\frac{f_3^2(r)}{f_3^2(r)+1}}} g_1(r) - 32\alpha b\beta B\left(n^2+4\right) r^2 f_3(r) f_1(r) g_1(r) \\
&- \left(\beta\left(2 f_1^{1/2}(r)-1\right) - 3 f_1^{1/2}(r) + 3\right) g_2(r) + \frac{8\alpha B r f_3(r)\left(f_1^{1/2}(r)-1\right)}{\frac{B(n f_2(r)+2br^2+2c) f_3(r)}{b(n^2+4)r} + A} \Big[2 + 4(\beta-1) f_1(r) + 8bB(\beta-1) \\
&\times \left(n^2+4\right) r^2 f_3(r) g_1(r) - (\beta+2) f_1^{1/2}(r) + 3\beta \Big] - \frac{8\alpha\left(\frac{bnr^2}{f_2(r)}+1\right) f_3^2(r)}{f_1(r)} \\
&\times \Big[\frac{4b^2(\beta-1) B^2\left(n^2+4\right)^2 r^4 f_3^2(r)\left(2 f_1^{1/2}(r)-3\right)}{\left(B\left(n f_2(r)+2br^2+2c\right) f_3(r) + Ab\left(n^2+4\right) r\right)^2} - \frac{2bB\left(n^2+4\right) r f_3^2(r)\left(f_1^{1/2}(r)-1\right)}{Ab\left(n^2+4\right) f_3(r) + B\left(n f_2(r)+2br^2+2c\right) f_3^2(r)} \\
&\times \left(2\beta\left(f_1^{1/2}(r)-4\right) - 3 f_1^{1/2}(r) + 9\right) - (5\beta-6)\left(f_1^{1/2}(r)-1\right)^2 \Big] - \beta r^4\phi f_1^2(r) + 2r^4\phi f_1^2(r)\Big],
\end{aligned} \tag{34}$$

where some new functions namely $f_i(r);\ i = 1, \ldots, 5$ and $g_i(r);\ i = 1,\ 2,\ 3$ have been introduced which are basically some complicated expressions of known terms and are defined as

$$f_1(r) = 1 + ar^2 e^{n\sin^{-1}(br^2+c)}, \quad f_2(r) = \sqrt{1-\left(br^2+c\right)^2}, \quad f_3(r) = \sqrt{ar^2 e^{n\sin^{-1}(br^2+c)}}, \tag{35}$$

$$f_4(r) = \sqrt{-\left(bR^2+c-1\right)\left(bR^2+c+1\right)}, \quad f_5(r) = \sqrt{-b^2R^4-2bcR^2-c^2+1}, \tag{36}$$

$$g_1(r) = \frac{\left(B\left(n\left(b^2r^4-c^2+1\right)+2\left(c-br^2\right)f_2(r)\right)f_3(r)+Ab\left(n^2+4\right)r\left(f_2(r)+bnr^2\right)\right)}{f_2(r)\left(B\left(nf_2(r)+2br^2+2c\right)f_3(r)+Ab\left(n^2+4\right)r\right)^2}, \tag{37}$$

$$g_2(r) = \frac{16\alpha b^2B^2\left(n^2+4\right)^2r^4f_3^2(r)\left(f_1^{1/2}(r)-1\right)}{\left(B\left(nf_2(r)+2br^2+2c\right)f_3(r)+Ab\left(n^2+4\right)r\right)^2}, \tag{38}$$

$$g_3(r) = \frac{8\alpha bB\left(n^2+4\right)rf_3^2(r)\left(f_1^{1/2}(r)-1\right)}{Ab\left(n^2+4\right)f_3(r)+Bf_3(r)(2c+2br^2+nf_2(r))}. \tag{39}$$

3. Matching of interior and exterior space–time geometries: Fixing the values of arbitrary constants

To describe a well established stellar interior, the matching procedure entails a crucial aspect. The main points of this grasp is to guarantee the continuity of the first and second fundamental forms across the junction interface defined by $\Sigma : r = R$. The inner manifold $\mathcal{M}^-$, in this case described by the pair (30)–(31), and the outer one $\mathcal{M}^+$ induce on Σ a metric tensor $g^-_{\mu\nu}$ and $g^+_{\mu\nu}$ respectively, describing the intrinsic geometric of the interface Σ. Then, the first fundamental form leads to

$$[ds^2]_\Sigma = 0, \tag{40}$$

or equivalently

$$e^{\chi-}|_{r=R} = e^{\chi+}|_{r=R} \quad \text{and} \quad e^{\zeta-}|_{r=R} = e^{\zeta+}|_{r=R}. \tag{41}$$

As we are dealing with a model without electric charges and cosmological constant contributions. Then, the exterior manifold $\mathcal{M}^+$ is described by the well-known vacuum Schwarzschild spacetime

$$ds^2 = \left(1-\frac{2M_{\rm Sch}}{R}\right)dt^2 - \left(1-\frac{2M_{\rm Sch}}{R}\right)^{-1}dr^2 - r^2\left(d\theta^2+\sin^2\theta d\phi^2\right). \tag{42}$$

So, putting together Eqs. (30)–(31), (41) and (42) one gets

$$1-\frac{2M_{\rm Sch}}{R} = \left(\frac{B\left(n\sqrt{1-\left(bR^2+c\right)^2}+2bR^2+2c\right)\sqrt{aR^2e^{n\sin^{-1}(bR^2+c)}}}{b\left(n^2+4\right)R}+A\right)^2, \tag{43}$$

$$\left(1-\frac{2M_{\rm Sch}}{R}\right)^{-1} = aR^2e^{n\sin^{-1}(bR^2+c)}+1, \tag{44}$$

where at the junction interface Σ the Schwarzschild mass $M_{\rm Sch}$ coincides with the total mass $m(R) = M$ contained by the fluid sphere. Now, the second fundamental form compromises the continuity of the extrinsic curvature tensor $K_{\mu\nu}$ across Σ. The continuity of the radial component of the extrinsic curvature tensor ı.e, $K^-_{rr} = K^+_{rr}$ assures a vanishing radial pressure at Σ, that is

$$p_r(R) = 0. \tag{45}$$

This fact is very important in the construction of compact objects, since (45) guarantees that the matter distribution contained by the fluid sphere in confined within the region $0 \le r \le R$, determining in this way the object size ı.e, its radius R. By solving the system of equations (43)–(44), we get the values of arbitrary constants a and A as follows

$$a = -\frac{2Me^{-n\sin^{-1}(bR^2+c)}}{R^2(2M-R)}, \tag{46}$$

$$A = \frac{bR\left(-2BRf_3(R)+n^2\sqrt{1-\frac{2M}{R}}+4\sqrt{1-\frac{2M}{R}}\right)-B\left(nf_4(R)+2c\right)f_3(R)}{b\left(n^2+4\right)R}, \tag{47}$$

and from Eqs. (33) and (45) one obtains

$$\alpha = -\frac{(\beta-2)R^6\phi(R-2M)\sqrt{1-\left(bR^2+c\right)^2}}{g(R)}, \tag{48}$$

where

$$g(R) = 16\left[2\beta M^3\left(\left(3-5\sqrt{\frac{R}{R-2M}}\right)\sqrt{-b^2R^4-2bcR^2-c^2+1}+4bnR^2\sqrt{\frac{R}{R-2M}}\right)-M^2R\right.$$
$$\times\left[\sqrt{-b^2R^4-2bcR^2-c^2+1}\left(\beta\left(12\sqrt{\frac{M}{R}}\sqrt{\frac{M}{R-2M}}-33\sqrt{\frac{R}{R-2M}}+23\right)+8\sqrt{\frac{R}{R-2M}}-2\right)\right.$$

Table 1
The values of constant parameters for the constructed model by using different values of R, n and B, along with $\beta = -25$, $b = 0.001$ [km^{-2}], $c = 0.001$ and $\phi = 2.036$ [km^{-2}].

Star	Observed mass $M_\odot$	Radius [km]	n	a [km^{-2}]	A	B [km^{-1}]	α	$\frac{p_{rc}}{\rho_c} < 1$
PSR J1614–2230	1.97	12.182	1.5	0.00257238	0.444817	0.0233421	$-2.17139 * 10^{-30}$	0.138213
LMXB 4U 1608–52	1.74	11.751	1.8	0.00237023	0.475168	0.0231551	$-2.22733 * 10^{-30}$	0.114631
Cen X-3	1.49	11.224	2.1	0.00219616	0.515361	0.0229539	$-2.29844 * 10^{-30}$	0.0387793
EXO 1785–248	1.3	10.775	2.2	0.00211597	0.544558	0.0227945	$-2.36296 * 10^{-30}$	0.0402779
SMC X - 1	1.04	10.067	2.5	0.0020195	0.585176	0.0225762	$-2.46315 * 10^{-30}$	0.048509

$$+ 8b\beta nR^2\left(\sqrt{\frac{M}{R}}\sqrt{\frac{M}{R-2M}}+\sqrt{\frac{R}{R-2M}}\right)\Bigg] + M\left[R^2\sqrt{-b^2R^4-2bcR^2-c^2+1}\left[\beta\left[10\sqrt{\frac{M}{R}}\sqrt{\frac{M}{R-2M}}\right.\right.\right.$$
$$\left.\left. - 28\sqrt{\frac{R}{R-2M}}+23\right] + 12\sqrt{\frac{R}{R-2M}}-8\right] + 2b\beta nR^4\left(4\sqrt{\frac{M}{R}}\sqrt{\frac{M}{R-2M}}+\sqrt{\frac{R}{R-2M}}\right)\Bigg]$$
$$- R^3\left[\sqrt{-b^2R^4-2bcR^2-c^2+1}\left(\beta\left(2\sqrt{\frac{M}{R}}\sqrt{\frac{M}{R-2M}}-7\sqrt{\frac{R}{R-2M}}+7\right)+4\left(\sqrt{\frac{R}{R-2M}}-1\right)\right)\right.$$
$$\left. + 2b\beta nR^2\sqrt{\frac{M}{R}}\sqrt{\frac{M}{R-2M}}\right]\Bigg].$$

It is worth mentioning that mass of the compact structure, also can be obtained by using the continuity of the angular components of the extrinsic curvature, that is, the continuity of $K_{\theta\theta}$ and $K_{\phi\phi}$. The conditions (46)–(48) are the necessary and sufficient conditions to determine some of the constant parameters characterizing the toy model. In Table 1 are placed the numerical values. These data was obtained by fixing n and B as free parameters. In addition we have taken mass M of some relativistic strange star candidates.

4. Some physical aspects of strange star model in $f(T,\tau)$ gravity

In this section, we shall describe some physical features of the strange stars like density and pressures, EoS parameters, energy conditions, TOV equation, causality condition, adiabatic index, redshift and mass function etc. on the basis of their graphical behavior. We shall determine the stability and significance of our proposed model.

4.1. Metric components and state variables

It is well-known that any model representing a realistic compact structure should satisfy some general requirements in order to be physically and mathematically feasible. These formalities concern a well defined geometric structure for all $r \in [0, R]$ and well behaved thermodynamic variables $\{\rho, p_r, p_t\}$ everywhere inside the collapsed configuration. To analyze the former, one needs to check the trend of the metric potentials (30) and (31) within the interval $[0, R]$ i.e, from the center to the surface of the compact structure. As it is depicted in the left panel of Fig. 1, the inner geometry described by the metric potentials (30)–(31) is free from physical and mathematical singularities everywhere within the object. Furthermore, the radial metric potential and the temporal one i.e, $e^{\chi(r)}|_{r=0} = 1$ and $e^{\zeta(r)}|_{r=0} > 0$ evaluated at the center of the star provide the correct results, what is more at the junction interface $r = R$ both coincide, that is, $e^{-\chi(r)} = e^{\zeta(r)}$. This shows that the matching condition process is well established.

Regarding the thermodynamic variables, the right panel of Fig. 1 displays the trend of the density ρ within the compact object. It is observed that this state parameter is positive defined and monotonically decreasing function with increasing radial coordinate r. This means that ρ has its maximum value attained at $r = 0$. As can be seen, as the mass of the object decreases the central density also decreases in magnitude. Moreover, the upper panels in Fig. 2 shown the radial pressure p_r (left panel) and the transverse pressure p_t (right panel), where it is clear that at $r = 0$ both quantities coincide and as one approaches the surface of the object they separate. This mismatch indicates that the matter distribution is anisotropic in nature. The signal that these local anisotropies are favorable in the construction of stellar objects (at least from the theoretical point of view) is: $p_t > p_r$, or equivalently $\Delta = p_t - p_r > 0$. This brings as a consequence two relevant facts, namely: (i) a positive anisotropy factor allows obtaining more compact stellar structures [58], (ii) the stability and hydrostatic equilibrium are substantially improved (for more details see the following sections), since $\Delta > 0$ introduces a repulsive gradient, which helps counteract the gravitational gradient. In the lower left panel of Fig. 2 it is appreciated that the density ($\frac{d\rho}{dr} = 0$) and pressures gradients ($\frac{dp_r}{dr} = 0$ and $\frac{dp_t}{dr} = 0$) are negatives. This corroborates the monotonically decreasing behavior of the state parameters $\{\rho, p_r, p_t\}$. Besides, the lower right panel in Fig. 2 shows the trend of the anisotropy factor. As can be observed, at $r = 0$ one has $\Delta = 0$ as a consequence of the equality $p_r = p_t$ at the center of the star, while $\Delta > 0$ at every point towards the surface. In Table 2 are displayed the central density, the surface density and the central pressure. Since the nuclear density saturation is of order of 10^{14} [g/cm^3] and in Table 2 it is appreciated that both the central and surface density are beyond this limit, it is evident that within the framework of $f(T,\tau)$ gravity theory, one can build more dense and compact object than in the GR picture.

Table 2
The numerical values of the central density, surface density, central pressure, adiabatic index and critical adiabatic index, by using different values of n and B, along with $\beta = -25$, $b = 0.001$ km^{-2}, $c = 0.001$ and $\phi = 2.036$ [km^{-2}].

Star	$\rho(0) * 10^{15}$ [g/cm^3]	$\rho(R) * 10^{15}$ [g/cm^3]	$p_r(0) * 10^{35}$ [dyne/cm^2]	Γ_r	Γ_{crit}
PSR J1416–2230	7.674997	4.496497	9.406281	1.933476	1.548881
4 U 1608–52	6.795468	4.468226	6.901296	2.153520	1.530698
Cen X-3	6.077967	4.443117	4.857780	2.446311	1.510277
EXO 1785–248	5.598387	4.427251	3.491886	2.535048	1.494147
SMC X - 1	5.147164	4.409376	2.206757	2.248983	1.471032

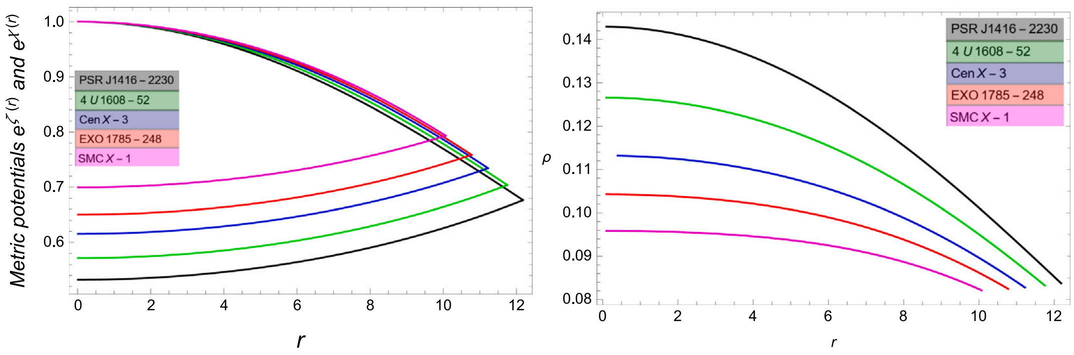

Fig. 1. Left Panel: The trend of metric potentials versus radial coordinate r. **Right Panel**: The behavior of the density against the radial coordinate r. These plots were obtained by using the numerical data given in Table 1.

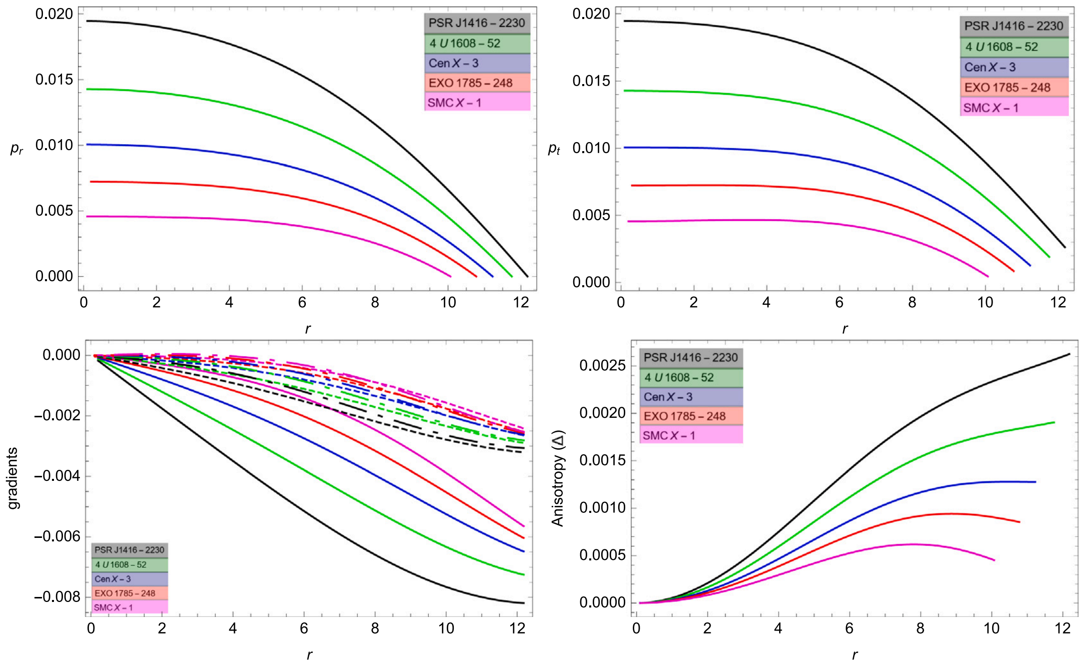

Fig. 2. Upper row: The radial pressure (left panel) and the tangential pressure (right panel), against the radial coordinate r. **Lower row:** The thermodynamic gradients (left panel) and the anisotropy factor versus the radial coordinate r (right panel). To build these plots the numerical data placed in Table 1 was employed.

To assure a well established matter distribution within the stellar interior, further analysis is required.[1] This analysis compromises the fulfillment of so-called Zel'dovich condition $\frac{p_{cr}}{\rho_{cr}} \leq 1$ [59] and the energy conditions [60]: The weak, null, dominant and strong

[1] Although the energy condition satisfaction only constitutes an heuristic analysis.

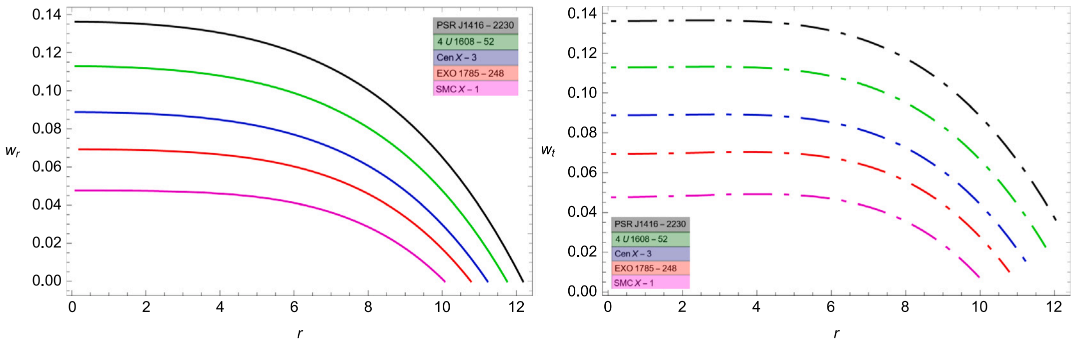

Fig. 3. The trend of EoS parameters versus radial coordinate r.

energy conditions. Those are given by the following inequalities

$$\text{WEC} : T_{\mu\nu}l^{\mu}l^{\nu} \geq 0 \text{ or } \rho \geq 0,\ \rho + p_i \geq 0 \tag{49}$$

$$\text{NEC} : T_{\mu\nu}t^{\mu}t^{\nu} \geq 0 \text{ or } \rho + p_i \geq 0 \tag{50}$$

$$\text{DEC} : T_{\mu\nu}l^{\mu}l^{\nu} \geq 0 \text{ or } \rho \geq |p_i| \tag{51}$$

$$\text{where } T_{\mu\nu}l^{\mu} \in \text{nonspace-like vector}$$

$$\text{SEC} : T_{\mu\nu}l^{\mu}l^{\nu} - \frac{1}{2}T^{\lambda}_{\lambda}l^{\sigma}l_{\sigma} \geq 0 \text{ or } \rho + \sum_i p_i \geq 0 \tag{52}$$

$$\text{TEC} : \rho - p_r - 2p_t \geq 0. \tag{53}$$

where $i \equiv (radial\ r, transverse\ t)$, l^{μ} and t^{μ} are time-like vector and null vector respectively. The Zel'dovich condition tells us that the speed of sound of the pressure waves cannot exceed the speed of light (this fact is related with the so-called causality condition, see below for further details). To check this feature, we have plotted the equation of state parameters (EoS) along the radial w_r and tangential w_t directions, respectively. These EoS parameters are defined by

$$w_r = \frac{p_r}{\rho}, \tag{54}$$

$$w_t = \frac{p_t}{\rho}. \tag{55}$$

The graphical illustration of these conditions is provided in Fig. 3 which indicates that these parameters are compatible with the inequalities: $w_r > 0$ and $0 < w_t < 1$ against the radial coordinate. Moreover, in the last column of Table 1 we have placed the numerical values for each case. This information corroborates the satisfaction of the Zeldovich condition. Regarding the energy conditions, one needs to taking into account that the matter distribution can be composed of a large number of fields. Therefore, it could be very complex to describe exactly the shape of the energy–momentum tensor. Thus, in order to have some impressions on the energy–momentum tensor behavior the above inequalities must be satisfied simultaneously. At this point it is worth noting that the energy conditions go beyond the idea that the energy must be positive defined. These conditions have a clear physical and geometric interpretation [60]. For example, the WEC implies that the energy density measured by an observer crossing a timelike curve is never negative. The SEC purports that the trace of the tidal tensor measured by the corresponding observers is always non-negative and finally DEC stand for mass–energy can never be observed to be flowing faster than light. Of course, the matter distribution driven the stellar interior of the present toy model, satisfies all these physical and geometric interpretations. It is worthy to mention here that since all quantities ρ, p_r and p_t exhibit positive behavior (as provided in Figs. 1 and 2), therefore all conditions will automatically satisfy except DEC and TEC [61]. For this reason, we check the validity of DEC and TEC only as shown in Fig. 4 which indicates that these inequalities are everywhere satisfied.

4.2. Stability via TOV equation

The equilibrium of a stellar configuration can be analyzed by the using an adapted Tolman–Oppenheimer–Volkoff (TOV) equation [62,63]. For the present stellar structure, the equilibrium depends upon three gradients which are F_a, the anisotropic gradient which is due to the presence of anisotropy in the fluid distribution, F_h termed as hydrostatic gradient and F_g, the gravitational gradient. Actually, both F_a and F_h gradients have a combine balancing effect equal to F_g. In case of $f(T,\tau)$ gravity there is an extra force F_e. Due to this balancing effect, the collapse of compact objects to a singularity point, during phenomena of gravitational collapse, may be avoided. So the presence of anisotropy within the stellar formation strengthens the equilibrium of stellar system. Mathematically, the adapted TOV equation, defining the equilibrium of the system, is given as

$$\frac{dp_r}{dr} + \frac{\zeta'(\rho + p_r)}{2} - \frac{2(p_t - p_r)}{r} + \frac{-1}{4\pi + \frac{\beta}{2}}\left(\frac{\beta\rho'}{4} - \frac{\beta p_r'}{4} - \beta p_t'\right) = 0, \quad F_g + F_h + F_a + F_e = 0,$$

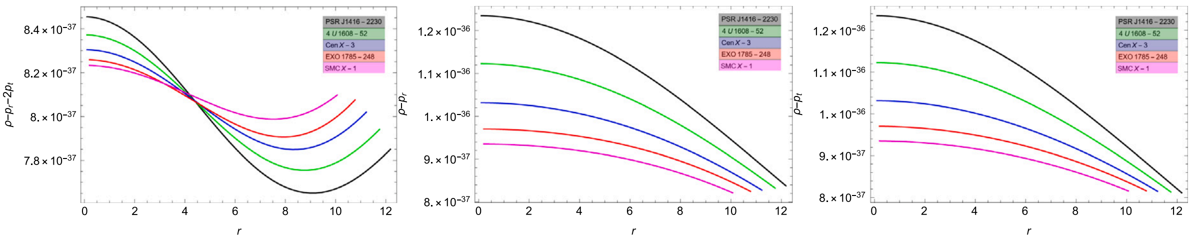

Fig. 4. The validity of energy conditions versus radial coordinate r. This plot was obtained by using the numerical displayed in Table 1.

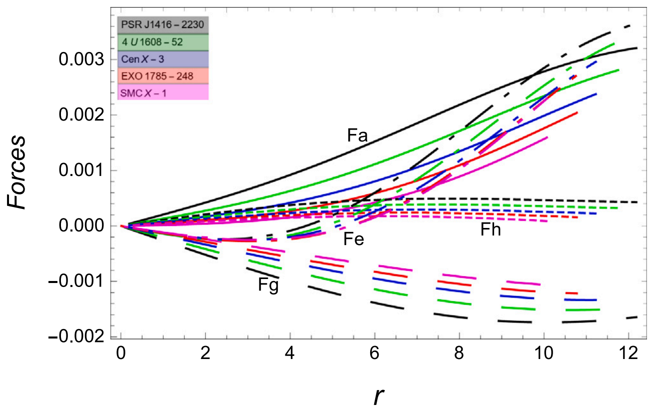

Fig. 5. The graphical illustration of the anisotropic F_a, gravitational F_g and hydrostatic F_h gradients versus radial coordinate r. These curves were built by using the values of constant parameters given in Table 1.

where

$$F_g = -\frac{\zeta'(\rho + p_r)}{2}, \quad F_h = -\frac{dp_r}{dr}, \quad F_a = \frac{2(p_t - p_r)}{r}, \quad F_e = \frac{-1}{4\pi + \frac{\beta}{2}}\left(\frac{\beta\rho'}{4} - \frac{\beta p_r'}{4} - \beta p_t'\right). \tag{56}$$

For the present stellar structures, it is seen from Fig. 5 that all these forces almost balance each other's effect by taking either small positive or negative values which authenticates the equilibrium of the constructed stellar system in this modified gravity.

4.3. Causality conditions

Here we shall talk about the consistency of causality conditions for the constructed stellar structure. It is argued that physically permissible and viable solution of dynamical equations for anisotropic fluid necessarily satisfy the causality conditions. Causality conditions are obtained by imposing some limitations on the radial and tangential sound velocities denoted by v_r^2 and v_t^2, respectively and is defined as $0 <| v_j^2 |< 1; \quad j = r, t$ (i.e., they should be less than the speed of light which is unity). These radial and transverse sound speeds can be defined as follows

$$v_r^2 = \frac{dp_r}{d\rho}, \quad v_t^2 = \frac{dp_t}{d\rho}.$$

It can be easily verified from their graphical illustration which is provided in Fig. 6 (left and middle panels) that both of the speeds fulfill the above requirement. It is interesting to mention here that Abreu et al. [64] discussed the stability of self-gravitating stellar sphere of anisotropic fluid by pointing out the stable region. This condition suggest that in potentially stable region, the radial speed v_r is greater than the transverse speed v_t, which means that the difference of two speeds must fulfill the inequality $|v_t^2 - v_r^2| \le 1$. The present stellar structure is also investigated under the light of this condition which is provided in Fig. 6 (right panel). It can be easily checked that this condition is also satisfied for our presented models.

5. Mass function, compactness factor, redshift, adiabatic index

Here we shall discuss some interesting parameters like mass function, compactness and redshift which confirm the physical existence of compact objects. The total mass is defined as follows

$$m(R) = 4\pi\int_0^R x^2\rho(x)dx, \quad u(r) = \frac{m(R)}{R}, \quad z_s = (1-2u)^{-\frac{1}{2}} - 1.$$

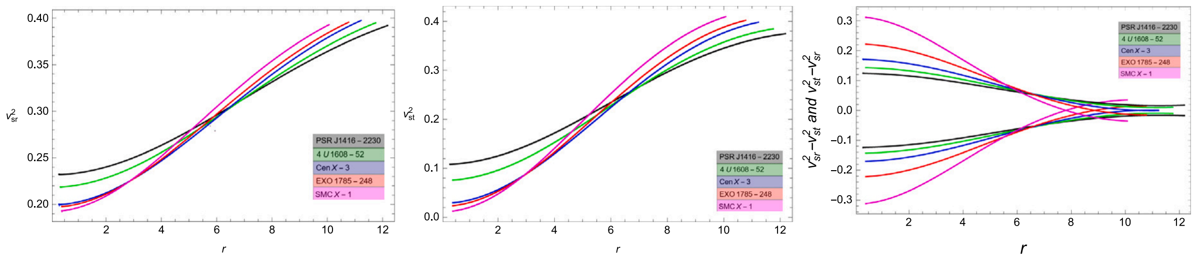

Fig. 6. Left panel and **Middle panel:** The squared radial and transverse sound speeds of the pressure waves versus the radial coordinate r. **Right panel**: The Abreu's stability criterion against the radial coordinate r. These plots were obtained by using the numerical data provided in Table 1.

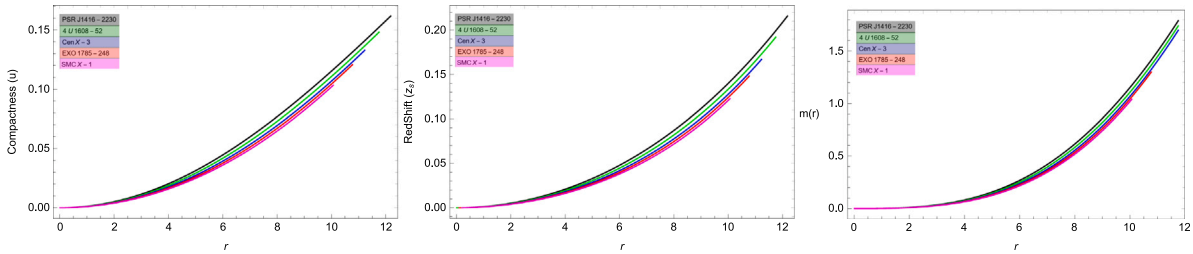

Fig. 7. Left panel: The compactness factor profile versus the radial coordinate r. **Middle panel:** The trend of the redshift against the radial coordinate r. **Right panel:** The mass function versus the radial coordinate r. The behavior of these quantities along the radial direction were constructed by employing the numerical data placed in Table 1.

However, one can also define the mass function $m(R)$ in a more convenient way by using the metric component $e^{\chi(r)}$ as follows

$$m(R) = \frac{R}{2}[1 - e^{-\chi(R)}].$$

It is mentioning that for a spherically symmetric and static configuration, the maximum limit of compactness (the ratio of mass to radius) must lie within the range $u = \frac{m}{r} < \frac{8}{9}$ (in the unit system $c = G = 1$). From the left panel of Fig. 7, it can be easily checked that the compactness parameter u is compatible with this defined range. It is argued that the existence of positive anisotropy limits the value of redshift as $Z_s < 5$, called Bohmer and Harko condition. From the middle graph of Fig. 7, it can be seen that the maximum redshift value is 0.20 and hence in accordance with this condition. Fig. 7 (right panel) shows the graphical illustration of mass function of the considered strange stars in this gravity. It is seen that mass function exhibits positive, increasing behavior and regular at the center and hence compatible with the original masses of the stars.

5.1. Adiabatic index and ratio of pressures

In this part, we shall discuss the behavior of adiabatic index and ratio of pressures to density graphically. Adiabatic index is an important property which ensures the stability of an anisotropic strange star configuration. It is argued that for physically stable model of compact star, the radial adiabatic index should be compatible with the range $\Gamma_r > \frac{4}{3}$. For a compact star model, the radial adiabatic index is defined as

$$\Gamma_r = \left(1 + \frac{\rho}{p_r}\right)\frac{dp_r}{d\rho}.$$

Its graphical behavior is provided in the right part of Fig. 8 which indicates that the presented model is consistent with this limit and hence represents a promising model of compact star within this modified gravity. Nevertheless, to assure a stable model following the relativistic adiabatic index point of view, further analysis is required. This is so because relativistic corrections to the adiabatic index Γ could introduce some instabilities inside the star [65,66]. To overcome this issue in [67] was proposed a more strict condition on the adiabatic index Γ. This condition claim the existence of a critical value for the adiabatic index Γ_{crit}. To have a stable structure, this critical value depends on the amplitude of the Lagrangian displacement from equilibrium and the compactness factor $u \equiv M/R$. The amplitude of the Lagrangian displacement is characterized by the parameter ζ, so taking particular a form of this parameter the critical relativistic adiabatic index is given by

$$\Gamma_{\text{crit}} = \frac{4}{3} + \frac{19}{21}u, \tag{57}$$

where the stability condition becomes $\Gamma \geq \Gamma_{\text{crit}}$. As can be appreciated in Table 2 (fifth and sixth columns) the condition $\Gamma \geq \Gamma_{\text{crit}}$ is always satisfied. Therefore, we can conclude that the present toy model is stable under local radial perturbations introduced by the relativistic corrections. Furthermore, we also investigate the behavior of ratio $\frac{p_r+2p_t}{\rho}$ graphically (see left panel of Fig. 8). It is

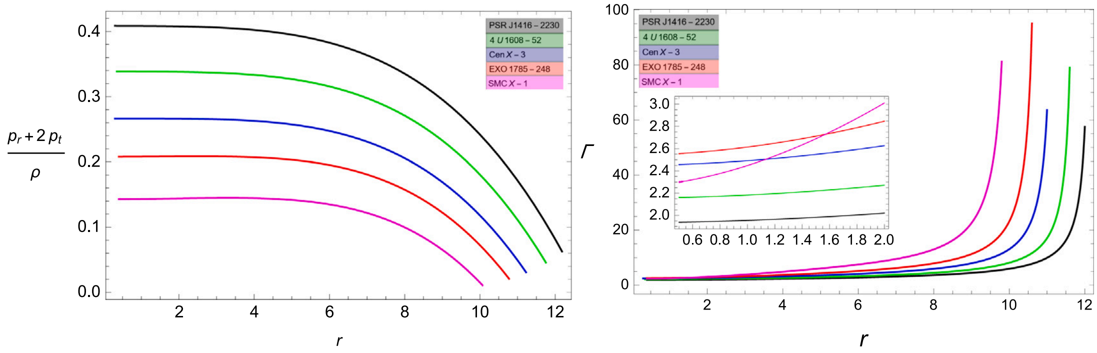

Fig. 8. Left panel: The behavior of ratio $\frac{p_r+2p_t}{\rho}$ against the redial coordinate r. **Right panel:** The adiabatic index trend versus radial coordinate r. To obtain these plots different values of the constants parameters exhibited in Table 1 were used.

argued that for physically viable compact star configuration, this ratio should exhibit positive decreasing behavior from center to outer boundary with maximum value at its center. In the considered gravity, this behavior of ratio can be easily verified from the left part of Fig. 8.

6. Conclusions

The construction of models representing the anisotropic compact stars using different candidates of modified gravity approach has became a center of research in modern cosmology. In literature, much work has been done on this subject and different significant models have been developed. In this work, we have studied the construction of compact structures exhibiting the static and spherically symmetric properties with matter distribution as anisotropic fluid in $f(T,\tau)$ gravitational framework. In literature [50,51], this topic has been discussed but the distinct feature of this study is the use of non-diagonal components for the formulation of basic structure as well as dynamical equations. For having viable solutions, one need to close the system of field equations. Since the present configuration involved 6 unknowns in set of 3 differential equations, therefore we have taken three valid assumptions into account. The use of Karmarkar condition for a spherically symmetric space–time has been declared as an easy and simple tool to obtain the solutions of field equations. By using the relationship suggested by Karmarkar condition, we have found both metric components and here we have also fixed the generic function $f(T,\tau)$ by considering a well-known model available in literature. In the following, we shall summarize the results obtained by exploring different physical features of the proposed models.

- It is seen that the presented structure is singularity free as the metric components showed regular at the center, positive and increasing behavior;
- The density and pressure functions exhibited positive decreasing behavior with maximum values at the center and hence referred to significant strange star structures;
- The density and pressure gradients have showed the appropriate negative decreasing behavior as suggested for the existence of such structures. Also, the Zeldovich's condition has been analyzed and found to be compatible with the inequality $\frac{p_{cr}}{\rho_{cr}} < 1$;
- It has been verified that the expression of energy conditions remain positive throughout the stellar configurations and hence suggesting valid star structures;
- As the presence of anisotropy throughout the matter distribution of star supports the anisotropic nature of the stars. In this work, the anisotropy parameter exhibited non-vanishing behavior, i.e., $\Delta > 0$ and hence authenticated physically stable and promising strange star structures in $f(T,\tau)$ gravity;
- The equilibrium and stability of the presented model has been confirmed graphically. It has been seen that three forces F_a, F_h and F_g balanced each other's effect and hence leaving the stellar configuration in an equilibrium state. Also, stability of the proposed stellar systems have been discussed by the analysis of causality conditions and Abreu's condition. It has been verified that our models are consistent with the limit of stability: $|v_{st}^2 - v_{sr}^2| < 1$;
- The radial and tangential EoS parameters showed compatibility with the suggested inequalities: $w_t > 0$ and $0 < w_r < 1$;
- It has been seen that the redshift, compactness and mass functions stayed positive and definite thorough this anisotropic matter distribution. It has been noticed that these parameters are in accordance with the some suggested limits for such configurations like Buchdahl and Bohmer and Harko condition;
- The adiabatic index exhibited positive increasing behavior and also satisfied the constraint $\Gamma_r > 4/3$. Thus indicated the existence of stable stellar structures in this gravity. The ratio of pressures and density has also been explored graphically indicating positive but decreasing behavior with maximum value at the center which is regarded as an essential condition for a realistic compact star.

Table 3
Summary of the achieved results using observed values of mass and radius of compact stars.

Expression	Result	Expression	Result
ρ	>0, satisfied	p_r	>0, satisfied
p_t	>0, satisfied	$\frac{p_{rc}}{\rho_c}$	≤ 1
Δ	>0, satisfied	$\frac{d\rho}{dr}$	<0, satisfied
$\frac{dp_r}{dr}$	<0, satisfied	$\frac{dp_t}{dr}$	<0, satisfied
F_a, F_h, F_g and F_e	balanced	$\rho+p_r$	>0, satisfied
$\rho+p_t$	>0, satisfied	$\rho-p_r$	>0, satisfied
$\rho-p_t$	>0, satisfied	$\rho+p_r+2p_t$	>0, satisfied
$m(r)$	>0, satisfied	$u(r)$	$0<u(r)<\frac{8}{9}$, satisfied
z_s	$0<z_s<5$, satisfied	w_r	$0<w_r<1$, satisfied
w_t	$0<w_r<1$, satisfied	v_r^2	$0<v_r^2<1$, satisfied
v_t^2	$0<v_r^2<1$, satisfied	$v_t^2-v_r^2$	$-1<\mid v_t^2-v_r^2\mid<1$, satisfied
Γ_r	$>\frac{4}{3}$, satisfied		

These results are also summarized in the form of Table 3. Finally, from our discussion, we can conclude that our proposed strange stellar models fulfill all the necessary requirements of a physically acceptable and admissible such structure. The chosen $f(T,\tau)$ model was firstly proposed by Harko et al. in a study [15] where they investigated the cosmological implications of this model by fixing both model parameters as $\alpha<0$ and $\beta<0$ and found viable results. In the present paper, we have explored the possibility of existence of compact stars in this gravitational framework by picking the same model along with similar negative choices of model parameters. It is found that the similar choices of free parameters also favor the existence of physically interesting stellar objects in this theory. It would be interesting to study other compact star models in the realm of $f(T,\tau)$ gravity and analyze their physically significance.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

M. Zubair thank the Higher Education Commission, Islamabad, Pakistan for its financial support under the NRPU project with grant number 7851/Balochistan/NRPU/R&D/HEC/2017. F. Tello-Ortiz acknowledges financial support by project ANT—1956 at Universidad de Antofagasta, Chile. F. Tello-Ortiz thanks the Ph.D. program Doctorado en Física mención en Física Matemática de la Universidad de Antofagasta for continuous support and encouragement.

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